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CYCLIC NUMBERS AND GAPS

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Abstract

A positive integer n is said to be *cyclic* if every group of size n is cyclic. Using Erdős's 1948 asymptotic formula for the counting function of cyclic numbers, we show that the mean $m(c,n)$ and the variance $v(c,n)$ of the first n cyclic numbers obey $\lim_{n\to\infty} v(c,n)/(m(c,n))^2 = 1/3$, illustrating Taylor's law of fluctuation scaling. Under the Riemann Hypothesis, we prove that, if c_n is the nth cyclic number, then $c_{n+1} - c_n = o\left(\sqrt{p_n}\log p_n\right) = o\left(\sqrt{n}\log^{3/2} n\right)$ as $n \to \infty$, which is a stronger analogue of Cramér's estimate for prime gaps. By analogy with Firoozbakht's conjecture for primes, we conjecture that $c_n^{1/n} > c_{n+1}^{1/(n+1)}$ for every positive integer n excluding $n = 1, 2, 3$ and 5; and that $c_n^{1/(n-1)} > c_{n+1}^{1/n}$ for every positive integer n excluding $n = 1$; and that for every positive integer k, there exists a positive integer $N(k)$ such that, for all $n > N(k)$, $c_n^{1/(n+k)} > c_{n+1}^{1/(n+k+1)}$.

1. Introduction

A *cyclic number* is a positive integer n such that there exists only one group of size n, up to isomorphism. Cyclic numbers (sequence A003277 in the On-Line Encyclopedia of Integer Sequences [11], OEIS) provide a group-theoretic generalization of prime numbers (OEIS A000040), since every prime number is a cyclic number. Szele [13] proved that a positive integer n is a cyclic number if and only if $\gcd(n, \varphi(n)) = 1$, where gcd is the greatest common divisor and $\varphi(n)$ is Euler's totient function (the number of positive integers up to n that are relatively prime to n).

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Because cyclic numbers generalize prime numbers, we aim to shed light on the relationships among cyclic and prime or composite numbers (OEIS A002808). First, we find the asymptotic behaviors of the mean and the variance of the first n cyclic numbers, prime cyclic numbers, and composite cyclic numbers. Second, for the gaps between cyclic numbers, we prove, under the Riemann Hypothesis, an analogue of an estimate for prime gaps due to Cramér [5]. Our concluding section proposes several conjectures inspired by Firoozbakht's conjecture about primes and prime gaps.

Cyclic numbers generalize prime numbers from both a group-theoretic perspective and a number-theoretic perspective. Hence, given an established or conjectured group-theoretic or number-theoretic property of primes, it is natural to ask whether a corresponding result holds for cyclic numbers, and vice-versa. For example, properties of cyclic numbers often involve modular equivalences: cyclic numbers are natural numbers n satisfying $\phi(n)^{\phi(n)} \equiv 1 \pmod{n}$; and Michon conjectured that any divisor of a Carmichael number (which is a composite number k such that $a^{k-1} \equiv 1 \pmod{k}$ for every a coprime to k: OEIS A002997) is odd and cyclic (OEIS A003277). Do such results and conjectures generalize to primes?

For any positive real x, let the number of primes that do not exceed x be

$$\pi(x) := \sum_{\substack{p \leq x \\ p \text{ prime}}} 1. \tag{1}$$

We say that $\pi(\cdot)$ is the counting function of the primes (OEIS A000720). The prime number theorem (PNT) states that

$$\pi(n) \sim \frac{n}{\log n}, \quad n \to \infty.$$

Hadamard [9] and de la Vallée Poussin [14] proved the PNT. Apostol [1] and Bordellès [2] give related background.

Let c_n be the nth cyclic number and let the sequence of cyclic numbers be $c := (c_n)_{n \in \mathbb{N}} = (1, 2, 3, 5, 7, 11, 13, 15, \ldots)$. By analogy with (1), we define the cyclic number counting function as the number of cyclic numbers that do not exceed x:

$$A(x) := \sum_{\substack{m \leq x \\ m \text{ cyclic}}} 1.$$

Let γ be the Euler–Mascheroni constant. In 1948, Erdős [6] proved that

$$A(n) \sim \frac{ne^{-\gamma}}{\log\log\log n}, \quad n \to \infty, \tag{2}$$

and Pollack [12] proved a general power-series expansion of $A(n)$ with the first three terms

$$A(n) \sim \frac{ne^{-\gamma}}{\log\log\log n}\left(1 - \frac{\gamma}{\log\log\log n} + \frac{\gamma^2 + \pi^2/12}{(\log\log\log n)^2} + \cdots\right), \quad n \to \infty.$$

From (2), it follows that

$$\begin{aligned} c_n &\sim e^{\gamma} n \log\log\log n, \quad (3)\\ \lim_{n\to\infty} \frac{c_{n+1}}{c_n} &= 1,\\ \lim_{n\to\infty} \frac{n}{c_{A(n)}} &= 1,\\ \lim_{n\to\infty} \frac{\log A(n)}{\log n} &= 1. \end{aligned}$$

Using MATLAB, we computed the 1,164,951 cyclic numbers less than 4×10^6, namely, 1, 2, 3, 5, 7, 11, 13, 15, 17, . . . , and their 1,164,950 gaps, namely, 1, 1, 2, 2, 4, 2, 2, 2, ... We then calculated the counting function of the cyclic numbers

$$\begin{aligned} A(n) &:= \#\{c_m \mid c_m \le n \text{ for } n = 1, \dots, 4 \times 10^6\}\\ &= \{1, 2, 3, 3, 4, 4, 5, 5, 5, 5, 6, 6, 7, 7, \dots, \}. \end{aligned}$$

For n as large as 4×10^6, the ratio of the counting function $A(n)$ to Erdős's asymptotic expression in (2) slightly exceeds 1/2, and the ratio of the counting function $A(n)$ to Pollack's asymptotic expression in (1) slightly exceeds 0.3. This slow convergence leads us to consider both statistical and number-theoretic methods to study the growth of cyclic numbers.

Less seems to be known about the composite cyclic numbers (15, 33, 35, 51, 65, 69, 77, 85, 87, 91, . . .) (all but the first element of the sequence of non-prime cyclic numbers, OEIS A050384) or the sequence (1, 1, 2, 2, 4, 2, 2, 2, 2, 4, . . .) of gaps between consecutive cyclic numbers (cf. OEIS A097884) or the sequence (18, 2, 16, 14, 4, 8, . . .) of gaps between consecutive composite cyclic numbers.

2. Asymptotic Moments, Variance Function, and Taylor's Law of Cyclic Numbers

Definition 1. For positive real x, let $pc(x)$ be the counting function of prime cyclic numbers. Let $cc(x)$ be the counting function of composite cyclic numbers.

Obviously $A(x) = pc(x) + cc(x) + 1$. Also, $pc(x) = \pi(x)$ as all prime numbers are cyclic numbers.

Lemma 1. *The asymptotic equivalences* $A(x) \sim cc(x) \sim xe^{-\gamma}/\log\log\log x$ *hold as* $x \to \infty$.

Proof. Because $\pi(x)/A(x) \sim (x/\log x)/(xe^{-\gamma}/\log\log\log x) \to 0$ as $x \to \infty$, we have $A(x) \sim \mathrm{cc}(x) \sim xe^{-\gamma}/\log\log\log x$. □

Define $\mathbb{R}_1 := [1, \infty)$. A function $R : \mathbb{R}_1 \to \mathbb{R}_1$ is defined to be *regularly varying (at infinity)* if it is positive, measurable on $\mathbb{R}_1$, and, for every $\lambda \in \mathbb{R}_+ := (0, \infty)$, the limit

$$g(\lambda) := \lim_{x\to\infty} \frac{R(\lambda x)}{R(x)} \in \mathbb{R}_+$$

exists and is finite and nonzero [3]. For every regularly varying function R, $g(\lambda)$ takes this power-law form for some $\rho \in \mathbb{R}$ (see [7]):

$$g(\lambda) = \lambda^\rho. \tag{4}$$

The exponent ρ in (4) is often called the *index* of the regularly varying function R.

Throughout, the kth moment of the first n cyclic numbers $\{c_1, \ldots, c_n\}$ is defined as

$$\mu'(c, n, k) := n^{-1}[c_1^k + \cdots + c_n^k].$$

Their mean is defined as

$$m(c, n) := n^{-1}[c_1 + \cdots + c_n] = \mu'(c, n, 1)$$

and their variance as

$$\begin{aligned} v(c, n) &:= (n-1)^{-1}[(c_1 - m(c,n))^2 + \cdots + (c_n - m(c,n))^2] \\ &= \frac{n}{n-1}(\mu'(c,n,2) - \mu'(c,n,1)^2). \end{aligned}$$

Theorem 1. *The counting functions $A(n)$, $\pi(x)$, $\mathrm{pc}(x)$, $\mathrm{cc}(x)$ are regularly varying with index 1. Hence the following statistical properties of cyclic numbers are equally valid for the prime cyclic numbers and the composite cyclic numbers.*

The relation

$$\lim_{n\to\infty} \frac{v(c,n)}{(m(c,n))^b} = \begin{cases} \infty, & \text{if } b < 2; \\ \frac{1}{3}, & \text{if } b = 2; \\ 0, & \text{if } b > 2 \end{cases} \tag{5}$$

holds. For all $k \in \mathbb{N}$, as $n \to \infty$,

$$\mu'(c, n, k) \sim \frac{c_n^k}{k+1}.$$

Moreover, the relations

$$m(c, n) = \mu'(c, n, 1) \sim \frac{c_n}{2}, \qquad v(c, n) \sim \frac{c_n^2}{12}$$

hold. For $j, k \in \mathbb{N}$, the jth and the kth moments are related asymptotically by

$$\mu'(c, n, k) \sim \frac{(j+1)^{k/j}}{k+1}(\mu'(c, n, j))^{k/j}.$$

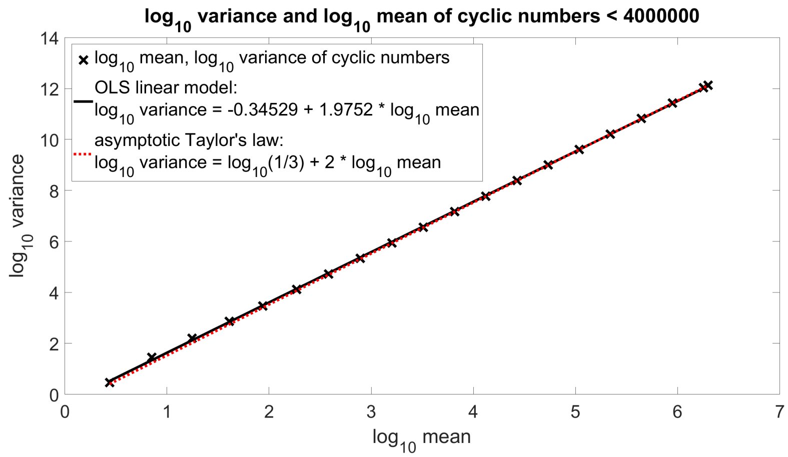

Figure 1: An illustration of the behavior of the variance and the mean for cyclic numbers.

In particular, for all $k \in \mathbb{N}$*, the moment* $\mu'(c,n,k)$ *asymptotically obeys a generalized TL (as defined in [8]):*

$$\mu'(c,n,k) \sim \frac{2^k}{1+k}(m(c,n))^k \text{ as } n \to \infty.$$

Proof. We calculate, for every $\lambda \in \mathbb{R}_+ := (0,\infty)$,

$$g(\lambda) := \lim_{n\to\infty} \frac{A(\lambda n)}{A(n)} = \lim_{n\to\infty} \frac{\frac{\lambda n e^{-\gamma}}{\log\log\log(\lambda n)}}{\frac{n e^{-\gamma}}{\log\log\log n}} = \lambda.$$

Hence $A(n)$ is regularly varying with index 1. The prime-counting function $\pi(x)$ was shown to be regularly varying with index 1 in [4]. The remainder of the theorem follows immediately from a more general result on integer sequences with regularly varying counting functions [3, Theorem 1]. □

Example 1. We illustrate Taylor's law for cyclic numbers (Figure 1), which plots the ordered pairs ($\log_{10}$ of the mean $m(c,n)$, $\log_{10}$ of the variance $v(c,n)$) of the first N cyclic numbers for $N = 2^2, 2^3, 2^4, \ldots, 2^{19}, 2^{20}, 1164951$ (solid black $\times$). According to (5), we should have (red dotted line) $v(c,n) \sim (1/3)m(c,n)^2$, hence $\log_{10} v(c,n) \sim \log_{10}(1/3) + 2 \cdot \log_{10} m(c,n)$, where $\log_{10}(1/3) \approx -0.4771$. Fitting a straight line to $(\log_{10} m(c,\ n), \log_{10} v(c,\ n))$ by ordinary least squares yields a very close approximation (solid black line). Evidently the mean and the variance of the first cyclic numbers converge rapidly to the power-law relationship posited by Taylor's law.

3. Gaps between Consecutive Cyclic Numbers

Let $g_n = p_{n+1} - p_n$ denote the nth prime gap. Cramér [5] proved that the Riemann Hypothesis (RH) implies that

$$g_n = O(\sqrt{p_n} \log p_n). \tag{6}$$

We prove an analogue of (6) (Equation (7) below) for cyclic number gaps. This result strengthens Cramér's result in that, for gaps of cyclic numbers, we use "little-o" in place of the "big-O" notation in Cramér's result for primes. This result further links cyclic and prime numbers.

Theorem 2. *For the first difference or gap $d_n := c_{n+1} - c_n$ of consecutive cyclic numbers, the RH implies*

$$d_n = o\left(\sqrt{p_n} \log p_n\right). \tag{7}$$

Proof. There are n cyclic numbers not exceeding c_n, by definition. Those cyclic numbers are either prime or non-prime. The non-prime cyclic numbers consist of the cyclic composite numbers and the integer 1. Define $\mathcal{J}(n)$ as the number of non-prime cyclic numbers not exceeding c_n. Also, we set $\mathcal{I}(n) := n - \mathcal{J}(n)$, giving us the number of prime cyclic numbers not exceeding c_n. Finally, define $\mathcal{C}_n$ as the nth non-prime cyclic number. Then

$$c_n = \begin{cases} p_{\mathcal{I}(n)}, & \text{if } c_n \text{ is prime;} \\ \mathcal{C}_{\mathcal{J}(n)}, & \text{if } c_n \text{ is non-prime.} \end{cases} \tag{8}$$

According to (8), the difference $c_{n+1} - c_n$ falls into one of four cases: (i) the gap $c_{n+1} - c_n$ is of the form $p_{\mathcal{I}(n+1)} - p_{\mathcal{I}(n)}$, so that $c_{n+1} - c_n = g_{\mathcal{I}(n)}$; (ii) the gap $c_{n+1} - c_n$ is of the form $p_{\mathcal{I}(n+1)} - \mathcal{C}_{\mathcal{J}(n)}$, so that $c_{n+1} - c_n < g_{\mathcal{I}(n)}$; (iii) the gap $c_{n+1} - c_n$ is of the form $\mathcal{C}_{\mathcal{J}(n+1)} - p_{\mathcal{I}(n)}$, so that $c_{n+1} - c_n < g_{\mathcal{I}(n)}$; or (iv) the gap $c_{n+1} - c_n$ is of the form $\mathcal{C}_{\mathcal{J}(n+1)} - \mathcal{C}_{\mathcal{J}(n)}$, in which case the integer $\mathcal{C}_{\mathcal{J}(n)}$ is composite, and, since the greatest prime number strictly less than $\mathcal{C}_{\mathcal{J}(n)}$ is $p_{\pi(\mathcal{C}_{\mathcal{J}(n)})}$, we have that $c_{n+1} - c_n < g_{\pi(\mathcal{C}_{\mathcal{J}(n)})}$.

The PNT and the asymptotic formula (2) due to Erdős [6] imply that, loosely speaking, for large real x there are far fewer prime cyclic numbers not exceeding x than non-prime cyclic numbers not exceeding x and that, more precisely, asymptotically

$$\#\{m \leq x : m \text{ is non-prime and cyclic}\} \sim \frac{xe^{-\gamma}}{\log \log \log x}. \tag{9}$$

The asymptotic equivalences in (3) and (9) then give $\mathcal{J}(n) \sim n$. Since $\mathcal{I}(n) = \mathcal{J}(n) - n$, the algebra of limits implies that

$$\lim_{n\to\infty} \frac{\mathcal{I}(n)}{n} = \lim_{n\to\infty} \frac{\mathcal{J}(n)}{n} - \lim_{n\to\infty} \frac{n}{n} = 0,$$

hence $\mathcal{I}(n) = o(n)$. The equivalence in (9) also gives that

$$\mathcal{C}_n \sim e^{\gamma} n \log\log\log n,$$

so that

$$\mathcal{C}_{\mathcal{J}(n)} \sim e^{\gamma} n \log\log\log n,$$

which implies, by the PNT, that

$$\pi\left(\mathcal{C}_{\mathcal{J}(n)}\right) \sim \frac{e^{\gamma} n \log\log\log n}{\log n}.$$

Hence

$$\lim_{n\to\infty} \frac{\pi\left(\mathcal{C}_{\mathcal{J}(n)}\right)}{n} = 0,$$
$$\pi\left(\mathcal{C}_{\mathcal{J}(n)}\right) = o(n).$$

Define

$$\mathcal{K}(n) := \begin{cases} \mathcal{I}(n), & \text{if } c_{n+1} - c_n = p_{\mathcal{I}(n+1)} - p_{\mathcal{I}(n)} \text{ or } p_{\mathcal{I}(n+1)} - \mathcal{C}_{\mathcal{J}(n)} \text{ or} \\ & \mathcal{C}_{\mathcal{J}(n+1)} - p_{\mathcal{I}(n)}; \\ \pi(\mathcal{C}_{\mathcal{J}(n)}), & \text{otherwise.} \end{cases}$$

Since $\mathcal{I}(n) = o(n)$ and $\pi(\mathcal{C}_{\mathcal{J}(n)}) = o(n)$, we have that

$$\mathcal{K}(n) = o(n).$$

Defining $f(n) := \sqrt{n}\log^{3/2} n$, we can, under the RH, rewrite Cramér's estimate in (6) as

$$g_n = O\left(f(n)\right).$$

Recalling our notation $d_n := c_{n+1} - c_n$ for cyclic gaps, we have shown that

$$d_n \le g_{\mathcal{K}(n)}.$$

So, Cramér's estimate, again under the RH, implies that

$$d_n = O\left(f(\mathcal{K}(n))\right).$$

Hence there are (absolute) constants M and n_0 such that for all $n \ge n_0$,

$$0 \le d_n \le M f(\mathcal{K}(n)).$$

Then

$$0 \le \frac{d_n}{f(n)} \le M \sqrt{\frac{\mathcal{K}(n)}{n}} \left(\frac{\log \mathcal{K}(n)}{\log n}\right)^{3/2}, \tag{10}$$

and $\mathcal{K}(n) = o(n)$ implies that

$$\lim_{n\to\infty} \sqrt{\frac{\mathcal{K}(n)}{n}} = 0.$$

Again since $\mathcal{K}(n) = o(n)$, this gives us that for all $\varepsilon > 0$ there exists N_ε such that for all $n \geq N_\varepsilon$, the bounds $0 < \frac{\log(\mathcal{K}(n))}{\log n} < 1 + \frac{\log \varepsilon}{\log n}$ hold, and we let $\varepsilon > 1$. So, using the above upper bound (10) for $d_n/f(n)$ as $n \to \infty$, we have $\sqrt{\mathcal{K}(n)/n} \to 0$, and $(\log \mathcal{K}(n)/(\log n))^{3/2}$ is eventually bounded above and below by finite values, i.e., so that

$$\lim_{n\to\infty} \frac{d_n}{f(n)} = 0,$$

or

$$d_n = o(f(n)),$$

which (by the PNT) is equivalent to $d_n = o\left(\sqrt{p_n} \log p_n\right)$. □

The $A(x)$ cyclic numbers that do not exceed positive real x specify $A(x) - 1$ gaps between consecutive cyclic numbers, so that, as a function of x, the mean of the gaps between consecutive cyclic numbers that do not exceed x is

$$\frac{d_1 + \cdots + d_{A(x)-1}}{A(x) - 1} = \frac{c_{A(x)} - c_1}{A(x) - 1} \sim \frac{c_{A(x)}}{A(x)} \sim e^\gamma \log\log\log A(x).$$

4. Conclusion

The equivalences summarized in Table 1, based on Theorem 1, illustrate some statistical relations among prime, cyclic, and composite numbers. Because the counting functions of prime, cyclic, and (it may easily be shown) composite numbers are all regularly varying with index 1, the first n entries of each sequence asymptotically obey the equivalent relations shown in the table. For future research, we introduce some open problems concerning cyclic numbers.

statistic	prime	composite	cyclic
mean	$\frac{p_n}{2}$	$\frac{\text{A002808}(n)}{2}$	$\frac{c_n}{2}$
kth moment	$\frac{p_n^k}{k+1}$	$\frac{\text{A002808}^k(n)}{k+1}$	$\frac{c_n^k}{k+1}$
variance	$\frac{p_n^2}{12}$	$\frac{\text{A002808}^2(n)}{12}$	$\frac{c_n^2}{12}$

Table 1: Statistics associated with prime, composite, and cyclic numbers.

In 1982, Firoozbakht conjectured that, if p_n is the nth prime starting from $p_1 = 2$, $p_2 = 3, \ldots$, then $(p_n)^{1/n}$ is a decreasing function of increasing $n = 1, 2, \ldots$.

Kourbatov [10] states a strict inequality

$$p_{k+1} < (p_k)^{1+1/k} \tag{11}$$

as equivalent to Firoozbakht's conjectured inequality, which therefore must assert that $(p_n)^{1/n}$ is a strictly decreasing function of increasing $n = 1, 2, \ldots$. Kourbatov shows that if Firoozbakht's conjecture is true, then $p_{k+1} - p_k < (\log p_k)^2 - \log p_k - 1$ for all $k > 9$ ($p_k \geq 29$). Thus, Firoozbakht's conjecture is potentially informative about prime gaps. Kourbatov also shows that, conversely, if $p_{k+1} - p_k < (\log p_k)^2 - \log p_k - 1.17$ for all $k > 9$, then Firoozbakht's conjecture is true. In his numerical studies, Firoozbakht's conjecture holds for all primes $p_k < 4 \times 10^{18}$.

Kourbatov [10] noted that (11) would imply Cramér's conjecture $g_n = O(\log^2 p_k)$, which is much stronger than Cramér's result (6) under the RH. We have proved an analogue for cyclic gaps of (6). As all prime numbers are cyclic numbers, we propose three conjectures of Firoozbakht type for cyclic numbers $c_1 = 1, c_2 = 2, c_3 = 3, c_4 = 5, \ldots$.

Conjecture 1. For every positive integer n excluding $n = 1, 2, 3$ and 5,

$$c_n^{1/n} > c_{n+1}^{1/(n+1)}.$$

The four exceptions are $1 < 2^{1/2}$, $2^{1/2} < 3^{1/3}$, $3^{1/3} < 5^{1/4}$, and $7^{1/5} < 11^{1/6}$.

Conjecture 2. For every positive integer n excluding $n = 1$,

$$c_n^{1/(n-1)} > c_{n+1}^{1/n}.$$

The only exception is $c_1 = 1 < c_2 = 2$.

Conjecture 3. For every $k \in \mathbb{N}$, there exists $N(k) \in \mathbb{N}$ such that, for all $n > N(k)$,

$$c_n^{1/(n+k)} > c_{n+1}^{1/(n+k+1)}.$$

In particular, if $k = 1$ or $k = 2$, then $N(k) = 5$; and if $k = 3$ or $k = 4$, then $N(k) = 11$.

We verified all three conjectures for the 1,164,951 cyclic numbers less than 4×10^6, and Conjecture 3 for $k = 1,\ 2,\ 3,\ 4$. Conjecture 1 strikes us as more surprising than Conjecture 2 because the first prime number is the second cyclic number, so for the primes among the cyclic numbers Conjecture 1 is stronger (gives tighter inequalities) than Firoozbakht's conjecture. For example, Conjecture 1 gives $c_6^{1/6} = 11^{1/6} \approx 1.4913 > c_7^{1/7} = 13^{1/7} \approx 1.4426$ whereas Firoozbakht's conjecture gives $p_5^{1/5} = 11^{1/5} \approx 1.6154 > p_6^{1/6} = 13^{1/6} \approx 1.5334$.

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