INTEGER SEQUENCES WITH REGULARLY VARYING COUNTING FUNCTIONS HAVE POWER-LAW VARIANCE FUNCTIONS

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Abstract

The variance function of an infinite increasing sequence of positive integers describes the variance of the first n elements of the sequence as a function of the mean of the first n elements. An asymptotic power-law variance function, sometimes called Taylor's law (TL) or fluctuation scaling, says that the variance is asymptotic to $c\times(\text{mean})^b$, for some real constants $c > 0$ and b. The counting function, evaluated at a positive real number y, is the number of elements of an infinite increasing sequence of natural numbers that are less than or equal to y. We show that when the counting function is regularly varying with positive index ρ, then TL holds asymptotically as n gets large with $c = 1/(\rho(\rho+2))$ and $b = 2$. Examples illustrate the theorem: integers raised to a fixed integer power, primes, the lesser of twin primes, primes in specified residue classes, prime constellations in the Bateman-Horn conjecture, primes from polynomials, perfect powers, triangular numbers, squares, pentagonal numbers, and some stochastically generated sequences.

– *In honor of my teacher, Anthony G. Oettinger (29 March 1929–26 July 2022)*

1. Introduction

Infinite increasing sequences of natural numbers (positive integers) provide an advantageous setting to study how the form and parameters of a variance function depend on observations. For a natural number n, define $m(a, n)$ as the average (or mean) and $v(a, n)$ as the variance of the first n members of an infinite sequence of natural numbers $a = (a(1), a(2), \ldots, a(n), \ldots)$, where $a(n) < a(n+1)$ for all n. If real constants $c > 0,\ b$ exist such that, as $n \to \infty$, $v(a, n) \sim c \cdot m(a, n)^b$, then we say that the sequence a has a *power-law variance function*, or obeys *Taylor's law (TL)*, or displays *fluctuation scaling*.

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Power-law variance functions have been widely and independently studied in ecology [6, 28, 29], mathematical statistics [30, 31, 32, 2, 3, 1], physics, earth sciences, and finance [15]. For brevity, we henceforth refer to a power-law variance function as Taylor's law or TL.

Because each integer sequence is defined mathematically, a number-theoretic setting is a controlled laboratory for understanding the conditions that lead to TL. Unlike observations from the real world, sampling error and measurement error are excluded from a mathematically defined integer sequence. Only the structure of the sequence remains. Each integer sequence is analogous to, but simpler than, a model organism in experimental biology.

Conversely, analyzing the variance function of increasing integer sequences can offer number theory fresh insights in addition to those provided by familiar diagnostic tools such as the natural density, the Schnirelmann density, and classifications of sequences by their mode of growth [18]. For example, for the prime numbers $\mathbb{P}$ (OEIS A000040), TL holds with $v(\mathbb{P}, n) \sim (1/3)m(\mathbb{P}, n)^2$ [10]. If the Hardy-Littlewood twin prime conjecture is true, TL holds with the same parameters $c = 1/3,\ b = 2$ for the lesser of twin primes (OEIS A001359) [10].

We show here that TL holds in both cases, and with identical parameters, not by some fluke, but as a consequence of general conditions under which an infinite increasing sequence of natural numbers obeys TL. Diverse (but not all) infinite increasing sequences of natural numbers obey these conditions and therefore obey TL. We give explicit formulas for c and b for this class of sequences. We illustrate this class with ten examples, including the primes and the lesser of twin primes. The following sections give definitions, results with proofs, and examples.

Prior work on connections between TL and number theory is limited, and mathematical analysis is rarer [22, 23]. M. Cohen [12] (no known relation) gave a different proof that the mean and variance of the primes obey TL. Demers [14] proved that the binomial coefficients on each row of Pascal's triangle (OEIS A007318) obey TL with $b = 2$.

2. Definitions and Background

2.1. Moments of Increasing Integer Sequences

The *natural numbers* are $\mathbb{N} := \{1, 2, 3, \ldots\}$. A basic sequence is the ordered natural numbers $(\mathbb{N}) := (1, 2, 3, \ldots)$. Let $\mathcal{A}$ be the set of all eventually strictly increasing infinite integer sequences $a : \mathbb{N} \to \mathbb{N}$ of the form $a := (a(1), a(2), \ldots)$ with $a(n) < a(n+1)$ for all sufficiently large $n \geq n_0(a) \geq 1$. We assume henceforth that $n_0(a) = 1$, i.e., that every $a \in \mathcal{A}$ is strictly increasing. This assumption entails no loss of generality because asymptotic properties of moments of a sequence are unaffected by a finite number of initial members of a.

For $a \in \mathcal{A},\ n \in \mathbb{N},\ k \in \mathbb{N}$, the kth *moment* of the elements of a up to and including $a(n)$ is

$$\mu'(a, n, k) := n^{-1}[(a(1))^k + \cdots + (a(n))^k]. \tag{1}$$

The *mean* of the elements of a up to and including $a(n)$ is

$$m(a, n) := n^{-1}[a(1) + \cdots + a(n)] = \mu'(a, n, 1). \tag{2}$$

The *variance* of the elements of a up to and including $a(n)$ is

$$v(a, n) := (n-1)^{-1}[(a(1) - m(a, n))^2 + \cdots + (a(n) - m(a, n))^2] \tag{3}$$
$$= \frac{n}{n-1}[\mu'(a, n, 2) - \mu'(a, n, 1)^2]. \tag{4}$$

If $f(x)$ and $g(x)$ are real-valued functions of real x and $g(x) > 0$ for all sufficiently large x, define $f(x) \sim g(x)$ to mean that $f(x)/g(x) \to 1$ as $x \to \infty$. In Riesel's [25, p. 61] cautious notation, define $f(x) \sim_c g(x)$ if $f(x) \sim g(x)$ is conjectured but not proved.

By definition, a sequence a obeys *Taylor's law (TL)* or, equivalently, has a *power-law variance function* if and only if, for some real constant b and some positive constant c, both depending on $a \in \mathcal{A}$,

$$v(a, n) \sim c \cdot m(a, n)^b \quad \text{or} \quad \lim_{n\to\infty} \frac{v(a, n)}{m(a, n)^b} = c. \tag{5}$$

Define $b(a, n)$, the *local elasticity* of the variance with respect to the mean for the sequence $a \in \mathcal{A}$ evaluated at n [9, p. 95, eq. (7)], as

$$b(a, n) := \frac{\log(v(a, n+1)/v(a, n))}{\log(m(a, n+1)/m(a, n))}. \tag{6}$$

TL, Equation (5), implies that

$$b = \lim_{n\to\infty} b(a, n). \tag{7}$$

2.2. Counting Functions, Exact and Asymptotic

For $a \in \mathcal{A},\ y \in \mathbb{R}_1 := [1, \infty)$, define the *counting function* $N(a, y) : \mathcal{A} \times \mathbb{R}_1 \to \mathbb{R}_1$ of a at y as the (integer) number of elements of a less than or equal to y, i.e., $N(a, y) := \max\{n \in \mathbb{N} \mid a(n) \le y\}$. Thus $N(a, a(n)) = n$. The faster $a(n)$ increases with n, that is, the sparser the successive elements $a(n)$ are among the natural numbers, the more slowly $N(a, y)$ increases with y. By assumption, if $a \in \mathcal{A}$, then $N(a, y) \to \infty$ as $y \to \infty$.

Let $\lfloor y \rfloor := \max\{n \in \mathbb{N} \mid n \le y\}$. If $a(n) = \lfloor y \rfloor$, then $N(a, y) = n$, and conversely. For all $a \in \mathcal{A},\ y \in \mathbb{R}_1,\ N((\mathbb{N}), y) = \lfloor y \rfloor \ge N(a, y)$ with equality if and only if

$a = (\mathbb{N})$. In some examples below, when a is clear from the context, we abbreviate the counting function $N(a, y)$ to $N(y)$.

An *asymptotic counting function* $\tilde{N}(a, y) : \mathcal{A} \times \mathbb{R}_1 \to \mathbb{R}_1$ of sequence a at $y \in \mathbb{R}_1$ is a non-decreasing function such that $\tilde{N}(a, y) \sim N(a, y),\ y \to \infty$. For example, $\tilde{N}((\mathbb{N}), y) := y \sim \lfloor y \rfloor,\ y \to \infty$. An asymptotic counting function is not unique. For example, both $y \sim \lfloor y \rfloor$ and $y + \log y \sim \lfloor y \rfloor$ are asymptotic counting functions of $(\mathbb{N})$.

2.3. Regularly and Slowly Varying Functions

A function $R : \mathbb{R}_1 \to \mathbb{R}_1$ is defined to be *regularly varying (at infinity)* if it is positive, measurable on $\mathbb{R}_1$, and, for every $\lambda \in \mathbb{R}_+ := (0, \infty)$, the limit

$$g(\lambda) := \lim_{x \to \infty} \frac{R(\lambda x)}{R(x)} \in \mathbb{R}_+ \tag{8}$$

exists and is finite and nonzero (see [16, pp. 275-284, 574-583], [5], and [26]).

Let $\rho \in \mathbb{R} := (-\infty, +\infty)$. Then $R(x) = x^\rho$ is regularly varying because, for all $\lambda \in \mathbb{R}_+$ and all $x \in \mathbb{R}_+$, we have $R(\lambda x)/R(x) = \lambda^\rho \in \mathbb{R}_+$. Thus in Equation (8), if $R(x) = x^\rho$, then $g(\lambda) = \lambda^\rho$. Conversely, for every regularly varying function R, $g(\lambda)$ takes this power-law form for some $\rho \in \mathbb{R}$ [16, pp. 275-284]:

$$g(\lambda) = \lambda^\rho. \tag{9}$$

The exponent ρ in (9) is often called the *index* of the regularly varying function R. Define $RV(\rho)$ to be the set of all regularly varying functions with index ρ.

A *slowly varying function* is a regularly varying function with index $\rho = 0$. Thus $RV(0)$ is the set of all slowly varying functions. A slowly varying function L satisfies, for every $\lambda \in \mathbb{R}_+,\ L(\lambda x) \sim L(x)$ or $g(\lambda) = 1$. A slowly varying function is a generalization of a constant function. Every regularly varying function with index ρ can be written as $R(x) = x^\rho L(x)$, where $L(x)$ is a slowly varying function.

For $a \in \mathcal{A},\ y \in \mathbb{R}_1,\ r \in (0, 1)$, define $F(a, y, r) := N(a, ry)/N(a, y)$. This ratio is the number of elements of a that do not exceed ry divided by the number of elements of a that do not exceed y. Thus $F(a, a(n), r) := N(a, ra(n))/n$. For fixed a and $y = a(n)$, $F(a, a(n), \cdot)$ is analogous to the empirical cumulative distribution function of the first n elements of a. For $a \in \mathcal{A},\ y \in \mathbb{R}_1,\ r \in (0, 1)$, if $N(a, y) \in RV(\rho)$, i.e., if $N(a, y)$ is regularly varying with index $\rho \in \mathbb{R}$, then

$$\lim_{y \to \infty} F(a, y, r) = r^\rho. \tag{10}$$

3. Results

3.1. Preliminary Lemmas

The cumulative mean and the cumulative variance increase without limit. For all strictly increasing $a \in \mathcal{A}$, $m(a,n)$ is monotonic increasing in n but $v(a,n)$ may not increase monotonically with n, depending on the spacing of the elements of a.

Lemma 1. *For all $a \in \mathcal{A}$, $k \in \mathbb{N}$, as $n \to \infty$, $\mu'(a,n,k) \to \infty$. In particular, $m(a,n) \to \infty$ and $v(a,n) \to \infty$.*

Proof. If $a = (\mathbb{N})$, then because $\sum_{j=1}^{n} j = n(n+1)/2$, we have $m((\mathbb{N}), n) = ((n+1))/2 \to \infty$ as $n \to \infty$. Also, because $\sum_{j=1}^{n} j^2 = (n(n+1)(2n+1))/6$, we have $v((\mathbb{N}), n) = (n/(n-1)) \cdot (\sum_{j=1}^{n} j^2/n - (m((\mathbb{N}), n))^2) = (n/(n-1)) \cdot (n^2-1)/12 \to \infty$ as $n \to \infty$. Now for any $a \in \mathcal{A}$, we have $a(n) \geq n$ for all $n \in \mathbb{N}$. Hence $m(a,n) := (1/n) \sum_{j=1}^{n} a(j) \geq m((\mathbb{N}), n)$. Further, for any $k \geq 1$, $(a(n))^k \geq n^k \geq n$ for all $n \in \mathbb{N}$. Hence $\mu'(a,n,k) \geq m((\mathbb{N}), n) \to \infty$.

Regarding the variance, the elements of a cannot be closer to one another than the natural numbers are, i.e., $|a(i) - a(j)| \geq |i - j|$, $i, j \in \mathbb{N}$. Therefore, by the formula for the variance based on Lagrange's identity [27, p. 39],

$$v(a,n) = \frac{n}{n-1} \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (a(i) - a(j))^2 \geq \frac{n}{n-1} \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (i-j)^2 = v((\mathbb{N}), n). \tag{11}$$

Hence $m(a,n) \to \infty$ and $v((\mathbb{N}), n) \to \infty$ as $n \to \infty$. □

The index of an infinite increasing integer sequence with a regularly varying counting function lies in $[0,1]$.

Lemma 2. *If $a \in \mathcal{A}$ has a regularly varying counting function $N(a,y)$, $y \in \mathbb{R}_1$, with index ρ, then $0 \leq \rho \leq 1$.*

Proof. By construction, $F(a,y,r) := N(a,ry)/N(a,y)$ must be at least weakly increasing in $r \in (0,1)$, so ρ cannot be negative. Also, $a(n) \geq n$, so $F(a,y,r)$ cannot increase faster than r, which entails $\rho \leq 1$. □

3.2. Main Result: A Regularly Varying Counting Function with Positive Index Implies Taylor's Law with Exponent $b = 2$

Theorem 1. *Let $a \in \mathcal{A}$ have a regularly varying asymptotic counting function $\tilde{N}(a,y)$ for $y \in \mathbb{R}_1 := [1,\infty)$, with positive index $\rho \in (0,1]$. Then the mean $m(a,n)$ and variance $v(a,n)$ of the first n elements $\{a(1), a(2), \ldots, a(n)\}$ obey*

$$\lim_{n\to\infty} \frac{v(a,n)}{(m(a,n))^b} = \begin{cases} \infty & \text{if } b < 2, \\ \frac{1}{\rho(\rho+2)} & \text{if } b = 2, \\ 0 & \text{if } b > 2. \end{cases} \tag{12}$$

For all $k \in \mathbb{N}$, as $n \to \infty$,

$$\mu'(a,n,k) \sim \frac{(a(n))^k \rho}{k+\rho}. \tag{13}$$

Therefore

$$m(a,n) \sim \frac{(a(n))\rho}{1+\rho} \tag{14}$$

and

$$v(a,n) = \frac{v(a,n)}{(m(a,n))^2}(m(a,n))^2 \sim \frac{1}{\rho(\rho+2)}\left(\frac{(a(n))\rho}{1+\rho}\right)^2 = \frac{(a(n))^2\rho}{(\rho+1)^2(\rho+2)}. \tag{15}$$

For $j,k \in \mathbb{N}$, the jth and the kth moments are related asymptotically by

$$\mu'(a,n,k) \sim \frac{\rho}{k+\rho}\left(\frac{j+\rho}{\rho}\right)^{k/j}(\mu'(a,n,j))^{k/j}. \tag{16}$$

In particular, for all $k \in \mathbb{N}$, the moment $\mu'(a,n,k)$ asymptotically obeys a generalized TL (as defined in [17]):

$$\mu'(a,n,k) \sim \frac{(\rho+1)^k}{\rho^{k-1}(\rho+k)}(m(a,n))^k \text{ as } n \to \infty. \tag{17}$$

Proof. Among the first n elements of $a \in \mathcal{A}$, the number of elements that exceed any integer $i \in [1, a(n)]$ is $n - N(a,i)$. Therefore, among the first n elements of $a \in \mathcal{A}$, the fraction of elements that exceed any integer $i \in [1, a(n)]$ is $(n - N(a,i))/n = 1 - N(a,i)/n$. Let $r_i := i/a(n)$. Then, by a standard formula for moments in terms of the survival function [21, 8],

$$\begin{aligned}
\mu'(a,n,k) &= 1 + \sum_{i=1}^{a(n)}[(i+1)^k - i^k]\left(1 - \frac{N(a,i)}{N(a,a(n))}\right) \\
&= 1 + \sum_{i=1}^{a(n)}[(i+1)^k - i^k]\left(1 - \frac{N(a,r_i a(n))}{N(a,a(n))}\right) \\
&= 1 + (a(n))^k \sum_{i=1}^{a(n)}[r_{i+1}^k - r_i^k]\left(1 - \frac{N(a,r_i a(n))}{N(a,a(n))}\right) \\
&\sim (a(n))^k \int_{r=0}^{1}(1 - r^\rho)d(r^k) \\
&= (a(n))^k k \int_{r=0}^{1} r^{k-1}(1 - r^\rho)dr = \frac{(a(n))^k\rho}{k+\rho}.
\end{aligned}$$

This proves Equation (13). Using Equation (13) for the jth and kth moments gives Equation (16). Putting $j = 1$ on the right side of Equation (16) gives Equation (17).

Now we prove Equation (12). In case $b = 2$, Equation (4) gives

$$\frac{v(a,n)}{m(a,n)^2} = \frac{n}{n-1}\left[\frac{\mu'(a,n,2)}{\mu'(a,n,1)^2} - 1\right]. \tag{18}$$

Therefore

$$\lim_{n\to\infty}\frac{v(a,n)}{m(a,n)^2} = \left(\lim_{n\to\infty}\frac{n}{n-1}\right)\left[\lim_{n\to\infty}\frac{\mu'(a,n,2)}{\mu'(a,n,1)^2} - 1\right] \tag{19}$$

$$= \lim_{n\to\infty}\left[\left(\frac{(a(n))^2\rho}{2+\rho}\right)\Big/\left(\frac{(a(n))^1\rho}{1+\rho}\right)^2\right] - 1 \tag{20}$$

$$= \frac{(1+\rho)^2}{\rho(2+\rho)} - 1 = \frac{1}{\rho(2+\rho)}, \tag{21}$$

which is TL, Equation (5), with exponent $b = 2$ and coefficient $c = [\rho(\rho+2)]^{-1}$.

In case $b \neq 2$, Equation (4) gives

$$\frac{v(a,n)}{m(a,n)^b} = \frac{n}{n-1}\left[\frac{\mu'(a,n,2)}{\mu'(a,n,1)^2}\mu'(a,n,1)^{2-b} - \mu'(a,n,1)^{2-b}\right] \tag{22}$$

$$= \frac{n}{n-1}\left[\frac{\mu'(a,n,2)}{\mu'(a,n,1)^2} - 1\right]\mu'(a,n,1)^{2-b}. \tag{23}$$

As $n \to \infty$, the first two factors on the right have finite limits equal to 1 and $c = [\rho(\rho+2)]^{-1}$, respectively, while the third factor on the right, $\mu'(a,n,1)^{2-b}$, approaches ∞ or 0 as $b < 2$ or $b > 2$. This proves Equation (12). □

4. Sequences with Regularly Varying Asymptotic Counting Functions with Positive Index

This section gives sequences in $\mathcal{A}$ that have regularly varying asymptotic counting functions with positive index ρ. By Theorem 1, these sequences obey

$$v(a,n) \sim [\rho(\rho+2)]^{-1}(m(a,n))^2. \tag{24}$$

Apart from the primes and the lesser of twin primes [10], these examples (summarized in Table 1) appear to be new instances of TL.

4.1. Integers Raised to a Fixed Power

First we calculate directly from the definition of the sequence a. Then we show that the result is given by Theorem 1.

For $h, k, n \in \mathbb{N}$, define $a(n) := hn^k$. Then $a(n) \in \mathbb{N}$ and $a := (h, h\cdot 2^k, h\cdot 3^k, \ldots) \in \mathcal{A}$. The mean of the first n elements $h, h\cdot 2^k, h\cdot 3^k, \ldots, h\cdot n^k$ of a is, asymptotically as $n \to \infty$,

$$m(a,n) \sim \frac{h}{n}\int_0^n t^k dt = \frac{h}{k+1}n^k = \frac{a(n)}{k+1}. \tag{25}$$

The second moment is, asymptotically,

$$\mu'(a,n,2) \sim \frac{h^2}{n}\int_0^n t^{2k} dt = \frac{h^2}{2k+1}n^{2k} = \frac{(a(n))^2}{2k+1}. \tag{26}$$

Therefore

$$\frac{v(a,n)}{(m(a,n))^2} \sim \frac{(k+1)^2}{2k+1} - 1 = \frac{k^2}{2k+1}. \tag{27}$$

To see that Equation (27) confirms Equation (24), we have that, since $a(n) := hn^k,\ n \in \mathbb{N}$, for $y \in \mathbb{R}_+$ the asymptotic counting function $\tilde{N}(a,y) = (y/h)^{1/k}$ is regularly varying with index $\rho = 1/k$, hence from Equation (24), $c = [\rho(\rho+2)]^{-1} = k^2/(2k+1)$ matches Equation (27).

The asymptotic behavior of the mean and variance of a is the same as that of any sequence $(f(1), f(2), f(3), \ldots, f(n), \ldots)$ generated by any polynomial of the form $f(n) := hn^k + h_1 n^{k-1} + \cdots + h_k n^0$ with $h > 0$ and any real coefficients $h_1, \ldots, h_k$ because, for large enough n, the leading term hn^k dominates the others.

4.2. Primes

Let the number of primes $\mathbb{P}$ that do not exceed $x > 0$ be $\pi(x) := \#\{p \in \mathbb{P} \mid p \le x\}$, where $\#\{\cdot\}$ denotes the number of distinct elements of the set $\{\cdot\}$. The prime number theorem states that

$$\pi(x) \sim \tilde{N}(\mathbb{P}, x) := \frac{x}{\log x}. \tag{28}$$

An equivalent statement of the prime number theorem is that $p_n \sim n\log n$, where p_n is the nth prime. Let $N(\mathbb{P}, p_n)$ be the exact counting function evaluated at the nth prime. It is easy and edifying to check that, as desired, $\tilde{N}(\mathbb{P}, p_n) \sim N(\mathbb{P}, p_n) = n$. Cohen [10] observed that, for any $r \in (0,1)$,

$$\lim_{x\to\infty} \frac{rx/\log(rx)}{x/\log(x)} = r. \tag{29}$$

We see, which Cohen [10] did not observe, that Equation (29) means that $\tilde{N}(\mathbb{P}, x) \in RV(1)$, i.e., $\tilde{N}(\mathbb{P}, x)$ is regularly varying with index $\rho = 1$. Hence Theorem 1 applies. As a result, the sequence of primes asymptotically obeys TL, Equation (24), with coefficient $c = 1/3$ and exponent $b = 2$, as [10, 12] proved directly. By Equation (17) and Equation (16), the higher moments of the primes also obey asymptotically a generalized TL with explicitly known coefficient and exponent.

4.3. Primes in Residue Classes

The reasoning given for primes applies immediately to the primes in individual residue classes. Crandall and Pomerance [13, p. 13] state the following "modern refinement" of a "marvelous (and nontrivial) theorem" of Dirichlet. The greatest common divisor (gcd) of two integers m and n is denoted $\gcd(m,n)$. Two integers m and n are said to be *relatively prime* or *coprime* [13] when $\gcd(m,n) = 1$. For any $d \in \mathbb{N}$, Euler's *totient function* $\varphi(d) := \#\{n \in \mathbb{N} \mid 1 \leq n \leq d,\ \gcd(n,d) = 1\}$ is defined as the number of natural numbers not exceeding d that are relatively prime to d. Let $\pi(x; d, n) := \#\{p \in \mathbb{P} \mid p \equiv n \bmod d,\ p \leq x\}$ be the exact counting function of primes in residue class $n \bmod d$ that do not exceed x. Then Dirichlet's theorem asserts that, for each $n \in \mathbb{N}$ such that $\gcd(n,d) = 1$,

$$\pi(x; d, n) \sim \tilde{N}(x) := \frac{1}{\varphi(d)} \cdot \frac{x}{\log x} \sim \frac{\pi(x)}{\varphi(d)} \text{ as } x \to \infty. \tag{30}$$

A simple numerical example illuminates the ideas. Let $x = 12$, $d = 6$. There are exactly 2 integers n such that $\gcd(n,6) = 1$, namely, $n = 1$ and $n = 5$, so $\varphi(6) = 2$. In the first case, when $n = 1$, $\pi(x; d, n) = \pi(12; 6, 1) = 1$ since 7 is the only prime not exceeding $x = 12$ in the residue class of $1 \bmod 6$. The first ten primes equal to $1 \bmod 6$ are 7, 13, 19, 31, 37, 43, 61, 67, 73, 79 (OEIS A002476). In the second case, when $n = 5$, $\pi(x; d, n) = \pi(12; 6, 5) = 2$ since 5 and 11 are the only primes not exceeding 12 in the residue class of $5 \bmod 6$. (The primes in the residue classes $2 \bmod 6$ and $3 \bmod 6$ are excluded since neither 2 nor 3 is relatively prime to 6.) Since exactly 5 primes do not exceed 12, Equation (30) would predict $\pi(12)/\varphi(6) = 5/2$ primes in each of the two residue classes $1 \bmod 6$ and $5 \bmod 6$.

As the asymptotic counting function $\tilde{N}(x)$ in Equation (30) is regularly varying with index 1, the number of primes in each of the $\varphi(d)$ residue classes modulo d asymptotically obeys TL with coefficient 1/3 and exponent 2, and also obeys a generalized TL with explicitly known coefficient and exponent, by Theorem 1.

4.4. Lesser of Twin Primes

A prime $p \in \mathbb{P}$ is the *lesser of twin primes* if $p + 2$ is also prime (OEIS A001359). Thus 3, 5, and 11 are the lesser of twin primes, but 7 is not since 9 is not prime. At least since Alphonse de Polignac in 1849, it has been conjectured, but is still not proved, that infinitely many twin primes exist. If $\pi_2(x)$ is the number of the lesser of twin primes that do not exceed $x \in \mathbb{R}_+$, Hardy and Littlewood [19] conjectured that $\pi_2(x) \sim_c Kx/(\log x)^2$ for a certain $K > 0$. Without using the concept of regular variation, [10] proved, in effect, that $Kx/(\log x)^2$ is regularly varying with index 1. Theorem 1 immediately implies TL with $c = 1/3$, $b = 2$ if the Hardy-Littlewood conjecture for twin primes is true.

4.5. Prime Constellations: Hardy and Littlewood (1923) Conjecture

We shall define a constellation, an admissible constellation, a prime constellation, and the generalized logarithmic integral, then state (in part) a conjecture of Hardy and Littlewood [19]. For any positive integer k, a finite-dimensional vector $A = (a_1, \ldots, a_k)$ of k integers in strictly increasing order such that $a_1 = 0 < \cdots < a_k$ is called a *constellation* of size k. For example, $A = (0)$ is a constellation of size 1 but $A = (3)$ is not a constellation because the first element is not 0. $A = (0, 2, 4)$ and $A = (0, 2, 6)$ are constellations of size 3.

For any prime p and any constellation $A = (a_1, \ldots, a_k)$, define

$$v(p, A) := \#\{a_1 \bmod p, \ldots, a_k \bmod p\}$$

to be the number of distinct residues mod p of the k elements of A. For example, if $A = (0, 2, 4)$ and $p = 3$, then $v(3, (0, 2, 4)) = \#\{0 \bmod 3,\ 2 \bmod 3,\ 4 \bmod 3\} = \#\{0, 2, 1\} = 3$, while if $A = (0, 2, 6)$ and $p = 3$, then $v(3, (0, 2, 6)) = \#\{0 \bmod 3,\ 2 \bmod 3,\ 6 \bmod 3\} = \#\{0, 2\} = 2$.

A constellation A is defined to be *admissible* if, for every prime p, we have $v(p, A) < p$. Obviously, if $p > k$, then $v(p, A) < p$ because there cannot be more residue classes than there are elements of A, so to determine whether a constellation A is admissible it suffices to check the primes $p \leq k$. In the previous examples, $A = (0, 2, 4)$ is not admissible because $v(3, (0, 2, 4)) = 3$ while $A = (0, 2, 6)$ is admissible because $v(3, (0, 2, 6)) = 2 < 3$ and $v(2, (0, 2, 6)) = 1 < 2$. The constellation (0, 2) is also admissible because $v(2, (0, 2)) = 1 < 2$.

If $A = (a_1, \ldots, a_k)$ is an admissible constellation, $n \geq 2$ is an integer and all elements of the vector $A + n := (n + a_1, \ldots, n + a_k)$ are simultaneously prime, we shall say that $A + n$ is a *prime constellation.* This definition implies that, if $A + n$ is a prime constellation, then $n + a_1 = n$ is prime so we can always write a prime constellation as $A + p$ for some admissible constellation A and some prime $p \in \mathbb{P}$. By this definition, (3, 5, 7) is not a prime constellation though all of its elements are primes because (0, 2, 4) is not admissible.

For real $x \geq 2$ and positive integer k, define the *generalized logarithmic integral*

$$Li(k, x) := \int_{t=2}^{x} \frac{dt}{(\log t)^k}. \tag{31}$$

Since $(\log t)^k/(\log t)^{k+1} \to 0$ as $t \to \infty$, $Li(k+1, x)/Li(k, x) \to 0$ as $x \to \infty$. Integrating by parts gives

$$Li(k, x) = \frac{x}{(\log x)^k} - \frac{2}{(\log 2)^k} + k \cdot Li(k+1, x) \sim \frac{x}{(\log x)^k} \text{ as } x \to \infty. \tag{32}$$

For a real $x \geq 2$ and an admissible constellation A, define $\mathbb{P}_A(x) := \{A + p \mid p \in \mathbb{P},\ p \leq x\}$ to be the set of prime constellations $A + p$ such that $p \leq x$. Define

$\pi_A(x) := \#\{\mathbb{P}_A(x)\}$ to be the number of prime constellations $A+p$ such that $p \le x$. For example, $\pi_{(0,2,6)}(10) = 1$ because $(0,\ 2,\ 6)$ is an admissible constellation and $5 + (0,\ 2,\ 6) = (5,\ 7,\ 11)$ are all simultaneously prime and there is no other such prime $p \le x = 10$.

Hardy and Littlewood [19] conjectured that, if $A = (a_1, \ldots, a_k)$ is an admissible constellation, then there exists a constant $C_A > 0$, dependent on A and independent of x, such that

$$\pi_A(x) \sim_c Q_A(x) := C_A \cdot Li(k, x). \tag{33}$$

The twin-prime conjecture that an infinity of twin primes exists is a consequence of this conjecture in the special case $A = (0, 2)$.

Since the asymptotic form of $Li(k, x)$ in Equation (32) is regularly varying with index $\rho = 1$, Theorem 1 implies that, if the Hardy-Littlewood conjecture is true, then asymptotically as $x \to \infty$, TL, Equation (24), holds with $b = 2,\ c = 1/3$ for the set of primes $p \le x$ such that $A + p$ is a prime constellation.

4.6. Primes from Polynomials: Bateman and Horn (1962) Conjecture

Bateman and Horn [4] proposed "a heuristic asymptotic formula" that is a far-reaching generalization of the conjecture Equation (33) of Hardy and Littlewood [19]. They supposed that $f_1, f_2, \ldots, f_k$ are polynomials in one variable with integer coefficients and positive leading coefficients. They supposed each polynomial $f_i,\ i = 1, \ldots, k$, is irreducible over the field of rational numbers and none is a constant multiple of another. Let $\pi(x, f_1, f_2, \ldots, f_k)$ denote the number of positive integers n between 2 and x inclusive such that $f_1(n),\ f_2(n), \ldots, f_k(n)$ are all primes, ignoring the finite number of values of n such that some $f_i(n) < 0$. (Bateman and Horn set the lower limit of n as 1, not 2, but the difference is immaterial for the asymptotic analysis. For consistency with the Hardy and Littlewood conjecture, it is more convenient for us to set the lower limit as 2.) Bateman and Horn conjectured that there exists a constant $C_B > 0$, dependent on $f_1,\ f_2, \ldots, f_k$ and independent of x, such that

$$\pi(x, f_1, f_2, \ldots, f_k) \sim_c Q(x, f_1, f_2, \ldots, f_k) := C_B \cdot Li(k, x). \tag{34}$$

The Hardy-Littlewood conjecture, Equation (33), is a special case in which $f_i(n) := n + a_i,\ i = 1, \ldots, k$, where $A = (a_1, \ldots, a_k)$ is an admissible constellation. If A is a constellation that is not admissible, the Bateman-Horn formula for C_B makes $C_B = 0$.

Since $Li(k, x)$ is asymptotically regularly varying with index $\rho = 1$ by Equation (32), Theorem 1 implies that, if the conjecture Equation (34) of Bateman and Horn [4] is true, then TL, Equation (24), holds asymptotically as $n \to \infty$ with $b = 2,\ c = 1/3$ for the set of positive integers n between 2 and x inclusive such that

$f_1(n), f_2(n), \ldots, f_k(n)$ are all primes, ignoring the finite number of values of n such that some $f_i(n) < 0$.

As an example, Bateman and Horn [4, p. 365] considered the pair of polynomials $f_1(n) = n,\ f_2(n) = n^2 + n + 1$ and counted the number $\pi(x, f_1, f_2)$ of primes $p \le x$ such that $p^2 + p + 1$ is also a prime for several values of $x \in [10,\ 113 \times 10^3]$.

The first three prime pairs of the form $(p, p^2 + p + 1)$ are $(2, 7),\ (3, 13),\ (5, 31)$ and there are no other examples with $p \le x = 10$. Bateman and Horn estimated that $C_B \approx 1.522/2 = 0.761$ in our notation.

Table 2 shows the number $\pi(x, f_1, f_2)$ of primes $p \le x$ such that $p^2 + p + 1$ is also a prime, for $x = 10^t,\ t = 1, 2, \ldots, 7$. For the values of x that appear in both Table 2 and the table of Bateman and Horn [4, p. 366], the counted numbers of such primes agree exactly. As x increases beyond 1000, the ratio variance/(mean)^2 decreases toward its proved asymptotic value 1/3.

4.7. Perfect Powers

The *perfect powers* (OEIS A072103) are all natural numbers of the form h^k, where $h, k \in \mathbb{N} - \{1\}$. The first ten perfect powers are 4, 8, 9, 16, 25, 27, 32, 36, 49, 64. Perfect powers that may be expressed in multiple different ways, such as $16 = 2^4 = 4^2$ or $81 = 3^4 = 9^2$, are listed only once. The perfect powers obey many beautiful theorems, including the Goldbach-Euler theorem and the Mihailescu theorem (formerly Catalan's conjecture). Nyblom [24] proved that an asymptotic counting function of the perfect powers is

$$\tilde{N}_{Nyb}(x) = x^{1/2} \in RV(1/2). \tag{35}$$

Jakimczuk [20] obtained more precise asymptotic counting functions such as

$$\tilde{N}_{Jak}(x) \sim x^{1/2} + x^{1/3} + x^{1/5} - x^{1/6} + x^{1/7} - x^{1/10} + g(x) \cdot x^{1/11} \tag{36}$$

where $\lim_{x\to\infty} g(x) = 1$. Because $\tilde{N}_{Nyb}(rx)/\tilde{N}_{Nyb}(x) = (rx)^{1/2}/x^{1/2} = r^{1/2}$, the perfect powers have an asymptotic counting function that is regularly varying with index $\rho = 1/2$. By Equation (14) and Equation (15), $m(a, n) \sim a(n)/3,\ v(a, n) \sim 4(a(n))^2/45$. By Theorem 1, the perfect powers (and the squares; see next example) asymptotically obey TL, Equation (24), with exponent $b = 2$ and coefficient $c = 4/5$.

Figure 1 illustrates Theorem 1 with $\rho = 1/2$ using the first $M = 100$ perfect powers. In Figure 1(a), the points are $(a(n), n/M),\ n = 1, 2, \ldots, M,\ a(100) = 6561$. The dashed (orange) curve plots $(a(n), \tilde{N}_{Nyb}(a(n))/\tilde{N}_{Nyb}(a(M))),\ n = 1, \ldots, M$, which is the asymptotic approximation based on Equation (35). The solid (orange) curve plots $(a(n), N_{Jak}(a(n))/N_{Jak}(a(M))),\ n = 1, \ldots, M$, which is the asymptotic approximation based on Equation (36). Both curves have no fitted parameters. In Figure 1(b), the points and curves are identical to those in (a) but the abscissa is on a logarithmic scale. In Figure 1(c), the points are $(\log_{10} m(a, n), \log_{10} v(a, n)),\ n = 1, \ldots, M$, which are asymptotically linear (TL) according to Theorem 1. The solid (orange) line plots $(\log_{10} m(a,n), \log_{10}(\frac{4}{5}) + 2\log_{10} m(a,n))$, $n = 1,\ldots,M$, which is the theoretical asymptotic form of TL. The solid line has no fitted parameters. In Figure 1(d), using $b(a,n)$ from Equation (6), the points are $(n, b(a,n))$, $n = 1,\ldots,M$. The solid (orange) horizontal line plots $(n,2)$, $n = 1,\ldots,M$, because the theoretical asymptotic slope of TL in this case is 2 according to Theorem 1. The solid line has no fitted parameters. The visually close agreement between the points calculated numerically from the perfect powers and the lines or curves calculated from Theorem 1 suggests that the asymptotic theory has high descriptive value in this example.

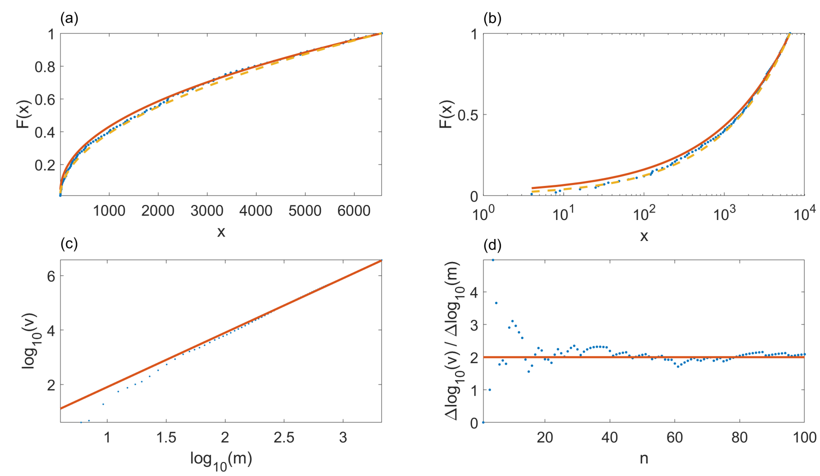

Figure 1: The first $M = 100$ perfect powers illustrate Theorem 1 with $\rho = 1/2$. For details, see text.

4.8. Triangular Numbers, Squares, Pentagonal Numbers

The *triangular numbers* $\mathbb{T}$ (OEIS A000217) are $T(n) := n(n+1)/2$, $n \in \mathbb{N}$. They are so named because $T(n) = 1 + 2 + \cdots + n$. The first ten triangular numbers are 1, 3, 6, 10, 15, 21, 28, 36, 45, 55. The counting function $N_{\mathbb{T}}(x)$ of the triangular numbers obeys $N_{\mathbb{T}}(n(n+1)/2) = n$. Solving $x = n(n+1)/2$ for n and setting aside the negative solution gives $n = ([1+8x]^{1/2} - 1)/2$. For general $x \in \mathbb{R}_+$, $N_{\mathbb{T}}(x) = \lfloor([1+8x]^{1/2} - 1)/2\rfloor \sim \tilde{N}_{\mathbb{T}}(x) := \sqrt{2x}$, which is, as in the previous example, regularly varying with index $\rho = 1/2$. We check that

$$\lim_{x\to\infty} \frac{T(\tilde{N}_{\mathbb{T}}(x))}{x} = 1, \qquad \lim_{x\to\infty} \frac{\tilde{N}_{\mathbb{T}}(T(n))}{n} = 1. \tag{37}$$

By Equation (14) and Equation (15), $m(\mathbb{T},n) \sim T(n)/3$, $v(\mathbb{T},n) \sim 4(T(n))^2/45$ for the first n triangular numbers. By Equation (24), TL holds asymptotically as

$n \to \infty$ with exponent $b = 2$ and coefficient $c = 4/5$.

For the *squares* (OEIS A000290), $a(n) := n^2,\ n \in \mathbb{N}$, and the *pentagonal numbers* (OEIS A000326), $a(n) := (3/2)(n^2 - n),\ n \in \mathbb{N}$, an identical line of reasoning leads to the same conclusion. The exact counting function of the pentagonal numbers is

$$N_a(x) = \left\lfloor \frac{1 + \sqrt{1 + 8x/3}}{2} \right\rfloor. \tag{38}$$

An asymptotic counting function is

$$\tilde{N}_a(x) = \sqrt{2x/3}. \tag{39}$$

More generally, let $a(n) : \mathbb{N} \to \mathbb{N}$ be any quadratic polynomial in which the quadratic term has a positive coefficient C. Asymptotically, $a(n)/n^2 \to C$, so the reasoning for triangular numbers, squares, and pentagonal numbers leads to $v(a, n) \sim (4/5)[m(a, n)]^2$.

4.9. A Random Sequence with Constant Probabilities

Fix $P \in (0, 1)$. A sequence $a \in \mathcal{A}$ is generated by a random process: at each $n \in \mathbb{N}$, a coin is tossed, independently of all other coin tosses. The probability that the coin comes up heads is P. If the coin comes up heads, that value of n is included in the sequence a; if tails, n is not included in a. By a strong law of large numbers [7, p. 119, Theorem 6.4] $N(a, y)/\lfloor y \rfloor$ converges almost surely to P. Thus almost surely $N(a, y) \sim P\lfloor y \rfloor \sim Py := \tilde{N}(a, y) \in RV(1)$, i.e., the asymptotic counting function $\tilde{N}(a, y)$ is regularly varying with index $\rho = 1$. Theorem 1 implies that TL holds almost surely with $c = 1/3,\ b = 2$. This conclusion is easily confirmed by direct calculation, which we omit. From direct calculation, the mean and variance of the elements of a up to and including the nth element $a(n)$ are asymptotically, almost surely,

$$m(a, n) \sim a(n)/2, \tag{40}$$

$$v(a, n) \sim (a(n))^2/12, \tag{41}$$

which are consistent with $\rho = 1$ in Equation (14) and Equation (15).

4.10. A Random Sequence with Pareto Probabilities of Index $\alpha \in (0, 1)$

Fix $\alpha \in (0, 1)$. A sequence $a \in \mathcal{A}$ is generated by a random process: at each $n \in \mathbb{N}$, a coin is tossed, independently of all other coin tosses. We assign probability $n^{-\alpha}$ to the event that the coin comes up heads and that value of n is included in the sequence a; with probability $1 - n^{-\alpha}$, the coin comes up tails and n is not included in a.

Then $E(N(a, \lfloor y \rfloor)) = \sum_{n=1}^{\lfloor y \rfloor} n^{-\alpha} \sim y^{1-\alpha}/(1-\alpha)$. The asymptotic expression on the right is strictly increasing in y if and only if $\alpha \in (0, 1)$, which we assume. Since $y^{1-\alpha}/(1-\alpha) \in RV(1-\alpha)$, Theorem 1 predicts asymptotically TL with $c = 1/((1-\alpha)(3-\alpha))$, $b = 2$. This conclusion is confirmed by direct calculation, which we omit. From direct calculation, the mean and variance of the elements of a up to and including the nth element $a(n)$ are asymptotically, almost surely,

$$m(a,n) \sim a(n) \cdot \frac{1-\alpha}{2-\alpha}, \tag{42}$$

$$v(a,n) \sim (a(n))^2 \cdot \frac{1-\alpha}{(3-\alpha)(2-\alpha)^2}, \tag{43}$$

which are consistent with $\rho = 1 - \alpha$ in Equation (14) and Equation (15). The behavior of the mean, variance, and variance function when $\alpha \geq 1$ will be described elsewhere.

5. Beyond Taylor's Law

If an asymptotic counting function of a sequence $a \in \mathcal{A}$ is not regularly varying with positive index, then the variance function of a need not obey TL. For example, asymptotically exponentially increasing sequences, such as the Fibonacci (OEIS A000045), Lucas (OEIS A000032), and Catalan (OEIS A000108) numbers, have slowly varying asymptotic counting functions of the form $\tilde{N}(a, y) = C \log y$ for various constants C. Their variance functions demand a generalization of TL [11]. For such sequences, the local elasticity $b(a, n)$ in Equation (6) converges to $b = 2$ as $y \to \infty$, as in Theorem 1. But, contrary to the conclusion of Theorem 1, $v(a, n)/m(a, n)^2$ is asymptotically proportional to $\log a(n)$ instead of converging to a constant. The variance functions of other sequences with other counting functions remain unexplored.

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Name of sequence a	$a(n),\ n \in \mathbb{N}$	$\tilde{N}(a,y)$	ρ	$m(a,n) \sim$	$v(a,n) \sim$	c
Integers to power	$hn^k,\ h,k \in \mathbb{N}$	$(y/h)^{1/k}$	$1/k$	$\frac{a(n)}{k+1}$	$\frac{k^2(a(n))^2}{(2k+1)(k+1)^2}$	$\frac{k^2}{2k+1}$
Primes	$p \in \mathbb{P}$	$\frac{y}{\log y}$	1	$\frac{a(n)}{2}$	$\frac{(a(n))^2}{12}$	$\frac{1}{3}$
Primes n mod d	$p \equiv n \bmod d$	$\frac{y}{\varphi(d)\cdot\log y}$	1	$\frac{a(n)}{2}$	$\frac{(a(n))^2}{12}$	$\frac{1}{3}$
Lesser of twin primes	$p \in P_2$	$\sim_c \frac{Ky}{(\log y)^2}$	1	$\frac{a(n)}{2}$	$\frac{(a(n))^2}{12}$	$\frac{1}{3}$
Prime constellations	A	$\sim_c C_A Li(k,y)$	1	$\frac{a(n)}{2}$	$\frac{(a(n))^2}{12}$	$\frac{1}{3}$
Polynomial primes	$f_1, f_2, \ldots, f_k$	$\sim_c C_B Li(k,y)$	1	$\frac{a(n)}{2}$	$\frac{(a(n))^2}{12}$	$\frac{1}{3}$
Perfect powers	$h^k,\ h,k > 1$	$y^{1/2}$	$1/2$	$\frac{a(n)}{3}$	$\frac{4(a(n))^2}{45}$	$\frac{4}{5}$
Triangular numbers	$n(n+1)/2$	$(2y)^{1/2}$	$1/2$	$\frac{a(n)}{3}$	$\frac{4(a(n))^2}{45}$	$\frac{4}{5}$
Constant prob. P	$\Pr\{n \in a\} = P$	Py	1	$\frac{a(n)}{2}$	$\frac{(a(n))^2}{12}$	$\frac{1}{3}$
Pareto $\alpha \in (0,1)$	$\Pr\{n \in a\} = n^{-\alpha}$	$\frac{y^{1-\alpha}}{1-\alpha}$	$1-\alpha$	$a(n)\frac{1-\alpha}{2-\alpha}$	$\frac{(a(n))^2(1-\alpha)}{(3-\alpha)(2-\alpha)^2}$	$\frac{1}{1-\alpha}\cdot\frac{1}{3-\alpha}$

Table 1: Sequence a with regularly varying asymptotic counting function $\tilde{N}(a,y)$ of index $\rho > 0$ obeys Taylor's law: $v(a,n) \sim c \cdot m(a,n)^b,\ b = 2,\ c = [\rho(\rho+2)]^{-1}$.

x	$\pi(x, f_1, f_2)$	minimal prime p	maximal prime p	mean	variance	variance / mean^2
10^1	3	2	5	3.3333E+00	1.5556E+00	0.140
10^2	8	2	89	3.5875E+01	1.0194E+03	0.792
10^3	23	2	911	3.7496E+02	1.1883E+05	0.845
10^4	117	2	9803	4.2562E+03	9.8014E+06	0.541
10^5	706	2	99923	4.4200E+04	9.0233E+08	0.462
10^6	4684	2	999773	4.5260E+05	8.9037E+10	0.435
10^7	33661	2	9999659	4.6220E+06	8.7156E+12	0.408

Table 2: The number $\pi(x, f_1, f_2)$ of primes $p \leq x$ such that $p^2 + p + 1$ is also a prime, for $x = 10^t,\ t = 1, 2, \ldots, 7$. x.xxxxE+yy equals x.xxxx $\times\ 10^{+yy}$.