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Generalizations of Bertrand's Postulate to Sums of Any Number of Primes

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In 1845, Bertrand conjectured what became known as Bertrand's postulate: twice any prime strictly exceeds the next prime [1]. Tchebichef, to use the spelling he used on the original publication, presented his proof of Bertrand's postulate to the Imperial Academy of St. Petersburg in 1850 and published it in 1852 [2]. It is now sometimes called the Bertrand-Chebyshev theorem.

Surprisingly, a stronger statement seems not to be well known, but is elementary to prove: the sum of any two consecutive primes strictly exceeds the next prime, except for the only equality, $2 + 3 = 5$. After I conjectured and proved this statement independently, a very helpful referee pointed out that Ishikawa published this result in 1934 (with a different proof) [3].

This observation is a special case of a much more general result, presented here as Theorem 1, that is also elementary to prove (given the prime number theorem), and perhaps not previously noticed: if p_n denotes the nth prime, and if $c_1, \ldots, c_g$ are nonnegative integers (not necessarily distinct), and $d_1, \ldots, d_h$ are positive integers (not necessarily distinct), and $g > h \geq 1$, then there exists a positive integer N such that

$$p_{n-c_1} + p_{n-c_2} + \cdots + p_{n-c_g} > p_{n+d_1} + \cdots + p_{n+d_h}$$

for all $n \geq N$. We prove this result using only the prime number theorem. We also sketch a way to find the least possible N for any specific instance of this result.

Bertrand's postulate can be expressed as $p_n + p_n \geq p_{n+1}$. This is the special case of Theorem 1 in which $g = 2$, $c_1 = c_2 = 0$, $h = 1$, $d_1 = 1$, and $N = 1$.

Additionally, we give a straightforward, independent proof of some other special cases, notably Theorem 2: for all $n > 1$, we have $p_{n-1} + p_n \geq p_{n+1}$ and equality holds only for $n = 2$. We give some numerical results and unanswered questions.

Main result

Theorem 1. *If $c_1, \ldots, c_g$ are $g > 1$ nonnegative integers (not necessarily distinct), and $d_1, \ldots, d_h$ are h positive integers (not necessarily distinct), with $1 \leq h < g$, then there exists a positive integer N such that, for all $n \geq N$,*

$$p_{n-c_1} + p_{n-c_2} + \cdots + p_{n-c_g} > p_{n+d_1} + \cdots + p_{n+d_h}.$$

Proof. For real-valued functions f, ϕ, each with real argument x, such that $\phi(x) > 0$ for all sufficiently large x, we define $f(x) \sim \phi(x)$ to mean that

$$\lim_{x\to\infty} \frac{f(x)}{\phi(x)} = 1.$$

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The prime number theorem says that if $x > 0$ and $\pi(x)$ is the number of primes that do not exceed x, then

$$\pi(x) \sim \frac{x}{\log x}.$$

Exactly n primes do not exceed the nth prime p_n, so as $n \to \infty$,

$$\begin{aligned}
\pi(p_n) = n &\sim \frac{p_n}{\log p_n}, \\
\log n &\sim \log p_n - \log\log p_n, \\
n \log n &\sim \frac{p_n}{\log p_n} \log p_n - \frac{p_n}{\log p_n} \log\log p_n = p_n \left(1 - \frac{\log\log p_n}{\log p_n}\right) \sim p_n.
\end{aligned}$$

In summary, $p_n \sim n \log n$. Therefore, for any fixed integer C such that $p_{n\pm C}$ is defined,

$$p_{n\pm C} \sim (n \pm C)\log(n \pm C) \sim n \log n.$$

It follows that

$$\begin{aligned}
p_{n-c_1} + p_{n-c_2} + \cdots + p_{n-c_g} &\sim gn \log n, \\
p_{n+d_1} + \cdots + p_{n+d_h} &\sim hn \log n.
\end{aligned}$$

Hence,

$$\frac{p_{n-c_1} + p_{n-c_2} + \cdots + p_{n-c_g}}{p_{n+d_1} + \cdots + p_{n+d_h}} \sim \frac{g}{h} > 1.$$

Therefore, there exists a positive integer N such that

$$p_{n-c_1} + p_{n-c_2} + \cdots + p_{n-c_g} > p_{n+d_1} + \cdots + p_{n+d_h}$$

for all $n \geq N$. ■

Finding the least N: an alternative approach

The same helpful referee suggested that, for greater specificity about N in Theorem 1, these inequalities could also be proved using inequalities of Rosser and Schoenfeld [5, (3.12), (3.13)]:

$$\begin{aligned}
n \log n &< p_n && \text{for } 1 \leq n, \\
p_n &< n(\log n + \log\log n) && \text{for } 6 \leq n.
\end{aligned}$$

We sketch the idea. Suppose we want to find the least positive integer N such that

$$p_n + p_{n-1} + p_{n-2} > p_{n+1} + p_{n+2} \qquad \text{for } n \geq N.$$

We look numerically for the least $n \geq 8$ such that

$$\begin{aligned}
n \log n + (n-1)\log(n-1) + (n-2)\log(n-2) > \\
(n+1)(\log(n+1) + \log\log(n+1)) + \\
(n+2)(\log(n+2) + \log\log(n+2)).
\end{aligned}$$

Thus, $n-2 \geq 6$, and the Rosser-Schoenfeld inequalities apply to all terms above. Such an n must exist because all three terms on the left side of the inequality are asymptotic to $n \log n$, but only two terms on the right side are. The remaining terms on the right side are asymptotically of smaller order of magnitude than $n \log n$ and therefore negligible. It turns out that the above inequality holds for $n = 33$. By the Rosser-Schoenfeld inequalities, it follows that

$$p_{33} + p_{32} + p_{31} > p_{34} + p_{35}.$$

For larger n, the inequality

$$p_n + p_{n-1} + p_{n-2} > p_{n+1} + p_{n+2}$$

must hold because the left side of the Rosser-Schoenfeld bounds

$$n \log n + (n-1)\log(n-1) + (n-2)\log(n-2)$$

grows faster than the right side

$$(n+1)(\log(n+1) + \log\log(n+1)) + (n+2)(\log(n+2) + \log\log(n+2)).$$

(But look out: $p_n + p_{n-1} + p_{n-2} - (p_{n+1} + p_{n+2})$ is neither weakly nor strictly increasing with increasing n, even for $n \geq 33$.) This $n = 33$ is higher than necessary because it is readily verified that

$$p_{10} + p_9 + p_8 = 71 > p_{11} + p_{12} = 68,$$

and that

$$p_n + p_{n-1} + p_{n-2} > p_{n+1} + p_{n+2}$$

holds for all n from 10 to 33. Since we have sketched the proof that the inequality must continue to hold for n larger than 33, we conclude that

$$p_n + p_{n-1} + p_{n-2} > p_{n+1} + p_{n+2}$$

for all $n \geq N = 10$.

Special cases

Theorem 2. *For all $n = 2, 3, \ldots$, we have $p_{n-1} + p_n \geq p_{n+1}$. Equality holds only for $n = 2$.*

We give a brief proof that is independent of Theorem 1.

Proof. Loo showed that for any integer $n \geq 3$, there is a prime in the interval $(n, 4(n+2)/3)$ [**4**, Corollary 2.2]. For $n = 1$ and $n = 2$, there is a prime in the interval $(n, 4(n+2)/3)$ because the corresponding intervals are $(1, 4)$ and $(2, 16/3)$. Since p_n is the smallest prime larger than p_{n-1},

$$p_n \in \left(p_{n-1}, \frac{4(p_{n-1}+2)}{3} \right).$$

Likewise,

$$\begin{aligned} p_{n+1} &\in \left(p_n, \frac{4(p_n+2)}{3} \right) \\ &\subset \left(p_n, \frac{4(4(p_{n-1}+2)/3+2)}{3} \right) = \left(p_n, \frac{16p_{n-1}+56}{9} \right). \end{aligned}$$

Now

$$\frac{16p_{n-1}+56}{9} \leq 2p_{n-1}$$

if and only if $28 \leq p_{n-1}$. So for all $p_{n-1} > 28$ we have

$$p_{n+1} < 2p_{n-1} < p_{n-1} + p_n.$$

To conclude the proof, we need only verify that $p_{n-1} + p_n \geq p_{n+1}$ for the primes p_{n-1} less than 28:

$$2+3=5, \quad 3+5>7, \quad 5+7>11, \quad 7+11>13, \quad 11+13>17,$$
$$13+17>19, \quad 17+19>23, \quad 19+23>29, \quad 23+29>31.$$

Since all primes p_n for $n > 1$ are odd, and the sum of two such primes is even, the strict inequality $p_{n-1} + p_n > p_{n+1}$ must hold for $n > 2$. ■

For any positive integers c, d (not necessarily distinct), define $N(c,d)$ to be the least finite positive integer, if it exists, such that $p_{n-c} + p_n \geq p_{n+d}$ for all $n \geq N(c,d)$. Theorem 1 implies that $N(c,d)$ always exists, and Theorem 2 implies that $N(1,1) = 2$. Using Shevelev et al.'s result that the list of integers k for which every interval $(kn, (k+1)n)$, $n > 1$, contains a prime includes $k = 1, 2, 3, 5, 9, 14$ and no other values of $k \leq 10^8$ [6] , we proved by extended calculations analogous to those in the proof of Theorem 2 that

$$N(2,2)=6, \quad N(3,3)=10, \quad N(4,4)=11, \quad N(5,5)=15.$$

Numerical results and open questions

Define

$$\delta(c,d,n) := p_{n-c} + p_n - p_{n+d}.$$

Then $\delta(c,d,n)$ is not always a monotonically increasing function of n even when $\delta(c,d,n) > 0$. For example,

$$\delta(2,2,6) = 7+13-19 = 1, \quad \delta(2,2,7) = 11+17-23 = 5,$$
$$\delta(2,2,8) = 13+19-29 = 3, \quad \text{and} \quad \delta(2,2,9) = 17+23-31 = 9.$$

We can also have successions of two or three (or perhaps more) identical values of $\delta(c,d,n)$ with given c, d and increasing n, for example,

$$\delta(1,1,50) = p_{49} + p_{50} - p_{51} = 223 = \delta(1,1,51) = p_{50} + p_{51} - p_{52}.$$

Is there any finite upper limit to the number of identical successive values of $\delta(c,d,n)$ with given c, d and increasing n?

For $c = 1, \ldots, 6$ and $d = 1, \ldots, 6$, we calculated $\delta(c,d,n)$ numerically for all primes less than 10^{10} and recorded the least n such that $p_{n-c} + p_n \geq p_{n+d}$ for that n and all larger observed values of n. We denoted this quantity by $M(c,d)$, and the results are displayed in Table 1). We distinguish the values $M(c,d)$ calculated numerically, using a finite (though large) set of primes, from the proved values $N(c,d)$. The first five diagonal elements $M(c,c)$, $c = 1, \ldots, 5$ are consistent with the values of $N(c,c)$ proved above. For $c < d$ in Table 1, we usually have $M(c,d) > M(d,c)$, e.g.,

$M(1, 3) = 6 > M(3, 1) = 5$, but not always, e.g., $M(5, 6) = 15 < M(6, 5) = 16$. In Table 1, and in much larger tables not reproduced here, we find that for a given c, $M(c, d)$ is monotonically weakly increasing with d, and for a given d, $M(c, d)$ is monotonically weakly increasing with c. Is this always true?

TABLE 1: The entry in the row labeled c and the column headed d is the numerically-calculated value $M(c, d)$ such that, for all $n \geq M(c, d)$, we have that $p_{n-c} + p_n \geq p_{n+d}$ among the 455,052,511 primes less than 10^{10}, and such that for $n = M(c, d) - 1$, $p_{n-c} + p_n < p_{n+d}$.

$c \downarrow, d \rightarrow$	1	2	3	4	5	6
1	2	5	6	9	10	11
2	3	6	8	10	10	12
3	5	7	10	10	12	13
4	5	9	10	11	13	14
5	7	10	11	12	15	15
6	7	11	12	14	16	16

The first 100 values of $M(1, d), d = 1, \ldots, 100$, are: 2 5 6 9 10 11 12 13 14 17 17 17 20 22 24 25 25 26 26 31 31 32 32 34 35 35 38 38 41 42 44 44 47 48 48 48 49 49 52 54 55 57 62 63 63 63 64 64 64 67 67 68 68 69 69 74 74 75 76 79 81 81 81 82 84 84 87 92 93 94 94 96 98 98 99 99 100 100 100 101 102 102 102 103 104 109 112 113 115 117 117 119 120 120 122 127 128 129 129 130. A search of the *On-Line Encyclopedia of Integer Sequences* [**7**, 2021-04-26] revealed no matching sequences.

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REFERENCES

[1] Bertrand, J. (1845). Mémoire sur le nombre de valeurs que peut prendre une fonction quand on y permute les lettres qu'elle renferme. *J. de l'Ecole Royale Polytechnique,* Cahier 30. 18: 123–140.

[2] Tchebichef, P. (1852). Mémoire sur les nombres premiers. *J. de mathématiques pures et appliquées,* series 1. 17: 366–390.

[3] Ishikawa, H. (1934). Ueber die Verteilung der Primzahlen. *Science Reports of the Tokyo Bunrika Daigaku, Sect. A* 2(31/32): 27–40.

[4] Loo, A. (2011). On the primes in the interval $[3n, 4n]$. *Int. J. Contemporary Mathematical Sciences* 6(37–40): 1871–1882.

[5] Rosser, J. B., Schoenfeld, L. (1962). Approximate formulas for some functions of prime numbers. *Illinois Journal of Mathematics* 6(1): 64–94. doi.org/10.1215/ijm/1255631807

[6] Shevelev, V., Greathouse, C. R., Moses, P. J. C. (2013). On intervals $(kn, (k+1)n)$ containing a prime for all $n > 1$. *J. Integer Sequences* 16, Article 13.7.3. pp. 1–14.

[7] OEIS Foundation Inc. (2019).The On-Line Encyclopedia of Integer Sequences. Online at: https://oeis.org.

Summary. In 1845, Bertrand conjectured what became known as Bertrand's postulate or the Bertrand-Chebyshev theorem: twice and prime strictly exceeds the next prime. Surprisingly, a stronger statement seems not to be well-known: the sum of any two consecutive primes strictly exceeds the next prime, except for the only equality $2 + 3 = 5$. Our main theorem is a much more general result, perhaps not previously noticed, that compares sums of any number of primes. We prove this result using only the prime number theorem. We also give some numerical results and unanswered questions.

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