



# Taylor's Law for Some Infinitely Divisible Probability Distributions from Population Models

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## Abstract

In a family of random variables, Taylor's law or Taylor's power law of fluctuation scaling is a variance function that gives the variance  $\sigma^2 > 0$  of a random variable (rv)  $X$  with expectation  $\mu > 0$  as a power of  $\mu$ :  $\sigma^2 = A\mu^b$  for finite real  $A > 0$ ,  $b$  that are the same for all rvs in the family. Equivalently, TL holds when  $\log \sigma^2 = a + b \log \mu$ ,  $a = \log A$ , for all rvs in some set. Here we analyze the possible values of the TL exponent  $b$  in five families of infinitely divisible two-parameter distributions and show how the values of  $b$  depend on the parameters of these distributions. The five families are Tweedie–Bar–Lev–Enis, negative binomial, compound Poisson–geometric, compound geometric–Poisson (or Pólya–Aeppli), and gamma distributions. These families arise frequently in empirical data and population models, and they are limit laws of Markov processes that we exhibit in each case.

**Keywords** Branching process · Compound distribution · Infinite divisibility · Markov process · Ornstein–Uhlenbeck process · Pólya–Aeppli distribution · Power law · Self-decomposability · Taylor's law · Taylor's power law · Tweedie family · Variance function

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## 1 Introduction

For practical purposes like pest control, conservation, and evaluating the yields of alternative agricultural treatments, agronomists, ecologists, and agricultural statisticians invented a concept today called the “variance function.” In a set of samples of some quantitative observations, the sample variance function describes how the sample variance varies from sample to sample as a function of the mean of each sample [7–9, 11].

Over several decades in the mid-twentieth century, multiple ecologists, apparently independently, observed that the log of the sample variance was approximated well by a linear function of the log of the sample mean, and that this model of the sample variance function was superior to several alternatives [11, 20, 22, 31]. The sample variance function that specifies the log sample variance as a linear function of the log sample mean has become known as Taylor’s law (TL), after the last ecologist who discovered it [31]. The mathematically equivalent statement that the sample variance is approximately proportional to some power of the sample mean has become known as Taylor’s power law, or fluctuation scaling in physical applications. Thousands of empirical examples of TL have been found in many different fields of science and finance [19, 32].

In theoretical studies of TL, a set of samples is modeled by a family (set) of random variables (rvs)  $\{X(\theta)\}_{\theta \in \Theta}$  indexed by some scalar- or vector-valued label  $\theta \in \Theta$  in an index set  $\Theta$  with at least two elements. The sample mean of empirical studies is replaced by the population mean  $0 < EX(\theta) := \mu(\theta) \leq \infty$ , and the sample variance of empirical studies is replaced by the population variance  $0 < \text{Var}X(\theta) := \sigma^2(\theta) \leq \infty$ . Then, for finite real  $a, b$  which may or may not depend on  $\theta$ , and for all  $\theta \in \Theta$ , TL asserts that  $\log \sigma^2(\theta) = a + b \log \mu(\theta)$  (log-linear form) or  $\sigma^2(\theta) = A[\mu(\theta)]^b$ ,  $A = \exp(a)$  (power form) or

$$b = \frac{\log \sigma^2(\theta) - a}{\log \mu(\theta)}, \quad \forall \theta \in \Theta. \quad (1)$$

The connections between the TL parameters  $a, b$  and the distributions of  $X(\theta)$  with moments  $\mu(\theta), \sigma^2(\theta)$  have long been of interest, and have sometimes been misunderstood. The best known example is the single-parameter family of Poisson( $\theta$ ) distributions with expectation  $\theta \in (0, \infty)$ , which are widely used as a model of pure randomness in integer counts. Here  $\sigma^2(\theta) = \mu(\theta) = \theta$ , so TL holds with  $a = 0$ ,  $b = 1$  for every  $\theta \in (0, \infty)$ . It does not follow that if TL holds with  $a = 0$ ,  $b = 1$ , then any of the distributions is Poisson( $\theta$ ) [15]. If  $a \neq 0$  or  $b \neq 1$ , then it is safe to conclude that at least one of the distributions is not a Poisson( $\theta$ ) distribution.

Another well known single-parameter family of distributions is the negative exponential family  $\text{Exp}(\theta)$ ,  $\theta \in (0, \infty)$ . Here  $\sigma^2(\theta) = [\mu(\theta)]^2$ , so TL holds with  $a = 0$ ,  $b = 2$ , regardless of  $\theta$ . Again, it does not follow that if TL holds with  $a = 0$ ,  $b = 2$ , then any of the distributions is  $\text{Exp}(\theta)$ .

Here we analyze the possible values of the TL exponent  $b$  in five families of infinitely divisible two-parameter distributions and show how the values of  $b$  depend on the parameters of these distributions. The shape of  $b(\theta)$  appears to be specific to each family. A rv  $X$  is defined to be infinitely divisible if and only if, for every positive integer  $n$ , there exist  $n$  independent and identically distributed (iid) rvs  $X_{n,1}, \dots, X_{n,n}$  such that  $X_{n,1} + \dots + X_{n,n} \stackrel{d}{=} X$ , where  $\stackrel{d}{=}$  means “has the same distribution as”. If  $X$  is infinitely divisible and  $E(e^{itX_{n,1}}) = E(e^{itX})^{1/n}$  is the characteristic function of (say)  $X_{n,1}$ , then for all  $t \in R$

$$\exp \left\{ -n \left( 1 - E(e^{itX_{n,1}}) \right) \right\} \rightarrow E(e^{itX}) \text{ as } n \rightarrow \infty,$$

showing that  $X$  is a weak limit of a compound Poisson sequence. More general iid sequences (of size  $k_n \rightarrow \infty$ ) converging weakly to  $X$  can be found in Theorem 5.2 of [30].

If  $X$  is infinitely divisible and satisfies Taylor’s power law with exponent  $b$ , then each of its constitutive summands  $X_{n,m}$  also satisfies Taylor’s power law with the same exponent  $b$ . To see this, suppose every rv in a family  $\{X(\theta)\}_{\theta \in \Theta}$ ,  $\Theta \neq \emptyset$ , satisfies

$$X(\theta) \stackrel{d}{=} X_{n,1}(\theta) + \dots + X_{n,n}(\theta), \quad n \geq 1, \tag{2}$$

where  $X_{n,1}(\theta), \dots, X_{n,n}(\theta)$  are iid as  $X_{n,1}(\theta)$ . Then  $X(\theta)$  satisfies TL (1) with exponent  $b$  if and only if  $X_{n,1}(\theta)$  satisfies TL (1) with the same exponent  $b$ . Indeed, taking the expectation of both sides of (2) gives  $\mu_X(\theta) = n\mu_{X_{n,1}}(\theta)$  and  $\sigma_X^2(\theta) = n\sigma_{X_{n,1}}^2(\theta)$ . If  $\sigma_X^2(\theta) = A[\mu_X(\theta)]^b$ , then  $n\sigma_{X_{n,1}}^2(\theta) = An^b(\mu_{X_{n,1}}(\theta))^b$  or  $\sigma_{X_{n,1}}^2(\theta) = An^{b-1}(\mu_{X_{n,1}}(\theta))^b$ , which is TL with the same exponent  $b$ . The coefficient  $A$  of  $X(\theta)$  leads to the coefficient  $A_n = An^{b-1}$  of  $X_{n,1}(\theta)$ . The converse is obvious.

The five families to be analyzed here are Tweedie–Bar–Lev–Enis, negative binomial, compound Poisson-geometric, compound geometric-Poisson (or Pólya–Aeppli), and gamma distributions. (Kendal [27] gives another interesting family.) These families arise frequently in empirical data and population models. They all have the special property that it is possible to express the dependence of the mean and variance on their parameters in such a way that  $a = 0$  or  $\sigma^2(\theta) = [\mu(\theta)]^b, \forall \theta \in \Theta$  or

$$b = \frac{\log \sigma^2(\theta)}{\log \mu(\theta)}, \quad \forall \theta \in \Theta. \tag{3}$$

Non-zero  $a$  can arise from the rescaling described in Sect. 7.

In addition to being infinitely divisible, some of these distributions are also self-decomposable (SD). A distribution or rv  $X$  is defined to be SD if, for every  $c \in (0, 1)$ , there exists an independent rv  $X_c$  such that

$$X(\theta) \stackrel{d}{=} c(\theta)X(\theta) + X_{c(\theta)}(\theta). \tag{4}$$

If  $X$  takes values in  $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ ,  $cX$  is to be interpreted as the  $c$ -Bernoulli thinning of  $X$ , which is defined as the sum of  $X$  iid Bernoulli rvs with success probability  $c$ .

If every rv in a family  $\{X(\theta)\}_{\theta \in \Theta}$ ,  $\Theta \neq \emptyset$ , satisfies TL (1) and (4), then the family  $\{X_c(\theta) \mid \theta \in \Theta\}$  also satisfies TL with the same value of  $b$  as the family  $\{X(\theta) \mid \theta \in \Theta\}$ ; and conversely. To see this, take the expectation of both sides of (4). Then  $\mu(\theta) = c(\theta)\mu(\theta) + EX_{c(\theta)}(\theta)$ , hence  $\mu(\theta) = EX_{c(\theta)}(\theta)/(1 - c(\theta))$ . Taking the variance of both sides of (4) gives  $\sigma^2(\theta) = c(\theta)^2 \cdot \sigma^2(\theta) + \text{Var}X_{c(\theta)}(\theta)$ , hence, using TL for the second equality,

$$\begin{aligned} \text{Var}X_{c(\theta)}(\theta) &= \sigma^2(\theta) \cdot (1 - c(\theta))^2 \\ &= A[\mu(\theta)]^b \cdot (1 - c(\theta))^2 \\ &= A[EX_{c(\theta)}(\theta)/(1 - c(\theta))]^b \cdot (1 - c(\theta))^2 \\ &= \{A(1 - c(\theta))^2(1 - c(\theta))^{-b}\}[EX_{c(\theta)}(\theta)]^b. \end{aligned}$$

This is TL with the same  $b$ , as claimed. The coefficient  $A$  of  $X(\theta)$  leads to the coefficient  $A(1 - c(\theta))^2(1 - c(\theta))^{-b}$  of  $X_{c(\theta)}(\theta)$ . The proof of the converse is routine, given the above equalities.

Every SD rv is unimodal. If  $X$  is SD and not discrete-valued, then it has a density. If  $X$  is SD and continuous, it is a weak limit of a continuous-time Lévy-driven Ornstein–Uhlenbeck

(OU) process. If  $X$  is SD and  $\mathbb{N}_0$ -valued, it is a weak limit of a pure-death branching process with immigration. See the Appendix.

Thus we consider three kinds of infinitely divisible distributions with two parameters. First, “bare” infinitely divisible distributions arise as specific limit laws, as we will describe. Second, infinitely divisible distributions that are compound Poisson geometric are the limit laws of immigration processes with total disasters. Third, infinitely divisible distributions that are SD are the limit laws of a pure-death branching process with immigration (if discrete) or of a Lévy-driven Ornstein–Uhlenbeck process (if continuous). In the latter two cases, there is a balance between events of birth (or immigration) and events of death, resulting in an equilibrium distribution. For some of the families under study, depending on the parameter range, the rv can switch from “bare” infinitely divisible to SD, perhaps suggesting some kind of phase transition. There may be other examples of infinitely divisible distributions that fall into these three categories, and it is of interest to know whether they would satisfy Taylor’s law.

## 2 Tweedie–Bar–Lev–Enis Distributions

The definition of Tweedie–Bar–Lev–Enis distributions is elaborate [3, 4, 23, 24] and will not be attempted here. An accessible expository account, with historical background, is [2]. For brevity, we will refer to these distributions and associated random variables as the TweBLE family.

With parameter  $\alpha \in [-\infty, 0) \cup \{0\} \cup (0, 1) \cup (1, 2) \cup \{2\}$  and

$$k(\theta) = \frac{1 - \alpha}{\alpha} \left( \frac{\theta}{1 - \alpha} \right)^\alpha \tag{5}$$

for the values of parameter  $\theta$  for which  $\left(\frac{\theta}{1-\alpha}\right)^\alpha$  is well-defined, the bilateral probability Laplace–Stieltjes transform (PLSt) of a TweBLE random variable (rv) [23, 24] is

$$\Phi(\lambda) = e^{-[k(\theta+\lambda)-k(\theta)]}.$$

The expression of  $k(\theta)$  is extended to cover the boundary cases  $\alpha = -\infty$  and  $\alpha = 0$ , respectively, by  $1 - e^{-\theta}$  and  $\log \theta$ . TweBLE rvs include Poisson ( $\alpha = -\infty$ ), compound Poisson-gamma ( $\alpha \in (-\infty, 0)$ ) (which is the sum of a Poisson-distributed number of iid gamma rvs all independent of the Poisson), negative exponential ( $\alpha = 0$ ), tempered one-sided stable ( $\alpha \in (0, 1)$ ), tempered two-sided extreme stable ( $\alpha \in (1, 2)$ ) and Gaussian ( $\alpha = 2$ ) rvs.

Considering (5), the log-Laplace transform (LLt) is

$$L(\lambda) := -\log \Phi(\lambda) = \frac{1 - \alpha}{\alpha} \left[ \left( \frac{\theta + \lambda}{1 - \alpha} \right)^\alpha - \left( \frac{\theta}{1 - \alpha} \right)^\alpha \right].$$

Hence the mean and variance of a TweBLE rv are

$$\begin{aligned} \mu &= L'(0) = \left( \frac{\theta}{1 - \alpha} \right)^{\alpha-1}, \\ \sigma^2 &= -L''(0) = \left( \frac{\theta}{1 - \alpha} \right)^{\alpha-2}. \end{aligned}$$

All TweBLE distributions satisfy  $\sigma^2 = \mu^b$ , which is TL with  $a = 0$ . Taylor’s law holds with exponent

$$b = \frac{\log \sigma^2}{\log \mu} = (2 - \alpha) / (1 - \alpha). \tag{6}$$

The extensions of TweBLE rvs to  $\alpha = -\infty$  and  $\alpha = 0$  must have  $b = 1$  and  $b = 2$ , respectively, to be consistent, and these values of  $b$  follow directly from the moments of Poisson and exponential distributions, respectively. Although a TweBLE rv has two parameters  $\alpha, \theta$ , the TL exponent  $b$  depends only on  $\alpha$ , unlike the two-parameter distributions in the following sections.

The graph of  $b$  versus  $\alpha$  when  $\alpha \in [0, 2]$  diverges at  $\alpha_c := 1$  and has two hyperbolic branches:

- one increasing from  $\alpha = -\infty$  to  $\alpha = 1^-$ , with  $b$  ranging from  $1^+$  (Poisson) to  $b = +\infty$ , through  $b = 2$  (gamma) when  $\alpha = 0$ .
- one increasing from  $\alpha = 1^+$  to  $\alpha = 2$ , with  $b$  ranging from  $b = -\infty$  to  $b = 0$  (Gaussian when  $b = 0, \alpha = 2$ ).

A TweBLE rv has a negative TL exponent  $b < 0$  only when its support is the whole real line, unlike some of the following examples, where  $b < 0$  can occur when the support is the nonnegative half line or the nonnegative integers.

No TweBLE distribution has  $b \in (0, 1)$  because there is no non-degenerate TweBLE PLSt with  $\theta > 2$ . Hence a population model that obeys TL with  $b \in (0, 1)$  cannot have the distribution of, and cannot be explained by using, a TweBLE rv. Examples of TL with  $b \in (0, 1)$  include a multiplicative population process in a Markovian environment [14,p. 33] and an infinity of examples with arbitrary distributions with finite means and finite variances [15,Example 3]. Therefore TweBLE distributions are not sufficient for modeling and understanding all instances of TL.

When  $\alpha < 1$ , TweBLE rvs are infinitely divisible because  $\Phi(\lambda) = \exp -L(\lambda)$  where  $L'(\lambda)$  is completely monotone, obeying  $(-1)^n L^{(n+1)}(\lambda) \geq 0$  for all  $\lambda > 0$ . When  $\alpha \in [1, 2]$ , TweBLE rvs are tempered  $\alpha$ -stable. In all cases, the Lévy jump measure can be found in [5]. Moreover, [5,Example 3] shows that they are SD as well except when  $\alpha \in (-\infty, 0)$ . For TweBLE rvs that are SD, the PLSt is

$$\Phi(\lambda) = \exp \left\{ \int_0^\lambda \frac{\log \Phi_0(\lambda')}{\lambda'} d\lambda' \right\} \tag{7}$$

for some PLSt  $\Phi_0(\lambda)$  of an infinitely divisible rv [30,Theorem 2.9, Eq. 2.12].

TweBLE distributions as defined in this Section belongs to the class of exponential families [2]. All such TweBLE distributions satisfy  $\sigma^2(\theta) = [\mu(\theta)]^b$ , which is TL with  $A = 1, a = 0$ , and  $b$  given by (6). Introducing the scaling (14) below, we switch to the class of exponential dispersion models, as defined by Jorgensen [24]. For this new class,  $\sigma^2(\theta) = A[\mu(\theta)]^b$  with  $A = e^a$  and  $a \neq 0$ , independent of  $b$ .

### 3 Negative Binomial Distributions

We shall show that the next two-parameter probability distribution families can be formulated in such a way that they also obey TL with the TweBLE family’s special property that  $a = 0$ ,

and we shall express  $b$  in terms of their parameters. As for TweBLE models, the scaling (14) will yield the version of TL with  $a \neq 0$ , independent of  $b$ .

With the parameters  $\alpha > 0, p \in (0, 1), q := 1 - p$ , a negative binomial rv  $X \stackrel{d}{=} NB(\alpha, p)$  has probability mass function

$$P[X = k] = \binom{k + \alpha - 1}{k} q^\alpha p^k, \quad k = 0, 1, 2, \dots$$

It has PLSt and LLt

$$\begin{aligned} \Phi(\lambda) &= \left( \frac{q}{1 - pe^{-\lambda}} \right)^\alpha, \\ L(\lambda) &= -\log \Phi(\lambda) = -\alpha \log q + \alpha \log(1 - pe^{-\lambda}), \\ L'(\lambda) &= \frac{\alpha pe^{-\lambda}}{1 - pe^{-\lambda}}, \end{aligned}$$

and mean and variance

$$\begin{aligned} \mu &= L'(0) = \frac{\alpha p}{q}, \\ \sigma^2 &= -L''(0) = \frac{\alpha p}{q^2} = \mu + \frac{1}{\alpha} \mu^2. \end{aligned} \tag{8}$$

Overdispersion holds, in the sense that, unlike the Poisson distribution, where the variance equals the mean, the negative binomial variance strictly exceeds the mean. The smaller  $\alpha$  is, the larger the variance compared to the mean. In ecology, overdispersion in population counts is sometimes interpreted as a resulting from heterogeneous field conditions (environmental variation making some places more favorable for individuals of the population than other places) or from aggregation or clustering (individuals behaving in a way that brings them near other individuals) [11, 31].

Taylor’s law holds with exponent

$$b = \frac{\log \sigma^2}{\log \mu} = 1 - \frac{\log q}{\log \left( \frac{\alpha p}{q} \right)}. \tag{9}$$

Then

$$\begin{aligned} b < 1 &\Leftrightarrow \frac{\alpha p}{q} < 1 \Leftrightarrow p < p_c := 1 / (1 + \alpha), \\ b < 0 &\Leftrightarrow \frac{\log q}{\log \left( \frac{\alpha p}{q} \right)} > 1 \Leftrightarrow q^2 + \alpha q - \alpha < 0 \\ &\Leftrightarrow q < q_0 := \frac{-\alpha + \sqrt{\alpha^2 + 4\alpha}}{2} \\ &\Leftrightarrow p > p_0 := \frac{2 + \alpha - \sqrt{\alpha^2 + 4\alpha}}{2} > 0. \end{aligned}$$

We always have

$$p_0 < p_c = \frac{1}{1 + \alpha}.$$

So for any fixed  $\alpha$ , the graph of  $b$  against  $p$  has two branches:

- one is concave and decreasing from  $b = 1$  to  $b = -\infty$  as  $p$  varies from  $0^+$  to  $p_c^-$  as a diverging point according as  $b \rightarrow -\infty$  as  $p \rightarrow p_c^-$ , passing through 0 as  $p = p_0$ .
- one is varying from  $b = +\infty$  as  $p \rightarrow p_c^+$  to  $b = 2$  as  $p \rightarrow 1^-$  possibly passing through a minimum  $b_{\min} > 1$  with  $b_{\min} < 2$  ( $= 2$ ) if and only if  $\alpha > 1$  (respectively  $\alpha \leq 1$ ). Hence

$$\begin{aligned} b &\in (0, 1) \text{ if } 0 < p < p_0, \\ b &\leq 0 \text{ if } p_0 \leq p < p_c, \\ b &> b_{\min} \text{ if } p > p_c. \end{aligned}$$

The range  $(1, b_{\min})$  for  $b$  is excluded.

If  $\alpha = 1$ , then  $b_{\min} = 2$  and  $b$  cannot fall in the interval  $(1, 2)$ . In this case, the negative binomial distribution reduces to the geometric distribution, and  $p_c = 1/2$  and  $p_0 = (3 - \sqrt{5})/2 = \phi^2$ , where  $\phi = (\sqrt{5} - 1)/2$  is a golden ratio.

Formula (9) identifies a manifold of  $(\alpha, p)$  points on which the TL exponent  $b$  is constant. To maintain a constant  $b$ , (9) requires both  $\alpha$  and  $p$  to vary. This result is surprising in light of the above negative binomial variance function (8),  $\sigma^2 = \mu + \alpha^{-1}\mu^2$ . For suppose  $\alpha$  is constant. Then as  $\mu \rightarrow 0$ ,  $\mu^2 \ll \mu$  and asymptotically  $\sigma^2$  is proportional to  $\mu$ , which is asymptotically TL with  $a = 0$ ,  $b = 1$ . However, as  $\mu \rightarrow \infty$ ,  $\mu^2 \gg \mu$  and asymptotically  $\sigma^2$  is proportional to  $\alpha^{-1}\mu^2$ , which is asymptotically TL with  $e^a = \alpha^{-1}$ ,  $b = 2$ . Moreover,  $\log \sigma^2$  is a strictly convex increasing function of  $\log \mu$  [16, Supp. Mat., lines 29–42], instead of a linear function as in TL. If the expectation  $\mu$  varies over only a small range in the data or the family of rvs, then  $\log \sigma^2$  can be closely approximated by (may be statistically indistinguishable from, in the case of empirical data) a locally linear function of  $\log \mu$  with a local slope that increases smoothly from 1 (for very small  $\mu$ ) to 2 (for very large  $\mu$ ). A wide range of means  $\mu$  would be required to display the curvature in the relation of  $\log \sigma^2$  to  $\log \mu$  when the underlying distributions are negative binomial with constant  $\alpha$ .

These observations illustrate that the form of the variance function of a two-parameter family of probability distributions depends strongly on how the parameters and their relation are constrained. We illustrate this point again in analyzing the gamma distribution below.

The negative binomial distribution is discrete and infinitely divisible (compound Poisson) because its probability generating function (pgf)  $\phi(z) := E(z^X)$  is

$$\phi(z) = \left(\frac{q}{1-pz}\right)^\alpha = \exp\{-[-\alpha \log q](1-c(z))\}, \tag{10}$$

where  $c(z) = \log(1-pz)/\log q$  is the pgf of a logarithmic series distribution for its clusters’ sizes.

The negative binomial distribution is also SD because its pgf may be written as (compare with (7) in the continuum)

$$\phi(z) = \exp\left\{-r \int_z^1 \frac{1-h(z')}{1-z'} dz'\right\}$$

for some rate  $r > 0$  and pgf  $h(z)$  such that  $h(0) = 0$  [30, Theorem 4.11, Eq. 4.13]. Indeed,  $r = \alpha p > 0$  and  $h(z) = qz/(1-pz)$  is a geometric pgf. See the Appendix.

When  $\alpha \in (0, 1)$ , then  $\pi := 1 - q^\alpha \in (0, 1)$  and the pgf  $\phi(z)$  in (10) equals

$$\phi(z) = \frac{1 - \pi}{1 - \pi\phi(z)},$$

where

$$\varphi(z) := \frac{1 - (1 - pz)^\alpha}{1 - q^\alpha}$$

satisfies  $\varphi(0) = 0$ . This  $\varphi(z)$  is a pgf because it is an absolutely monotone function: it obeys  $\varphi^{(n)}(z) \geq 0$  for all  $0 < z < 1$  and has positive coefficients in its power series expansion. Indeed, the coefficient of  $z^n$  in  $\varphi(z)$  is

$$\begin{aligned} [z^n]\varphi(z) &= \frac{p^n}{1 - q^\alpha} (-1)^{n-1} \frac{(\alpha)_n}{n!} \\ &= \frac{p^n}{1 - q^\alpha} \frac{\alpha[\bar{\alpha}]_{n-1}}{n!}, \quad n \geq 1 \end{aligned}$$

where  $(\alpha)_n := \Gamma(\alpha + 1) / \Gamma(\alpha + 1 - n) = \alpha(\alpha - 1) \cdots (\alpha - n + 1)$  is the falling factorial and  $[\bar{\alpha}]_n := \bar{\alpha}(\bar{\alpha} + 1) \cdots (\bar{\alpha} + n - 1)$ ,  $n \geq 1$ , is the rising factorial of  $\bar{\alpha}$  with  $[\bar{\alpha}]_0 := 1$ ,  $\bar{\alpha} = 1 - \alpha$ .

Because the negative binomial distribution is SD for all  $\alpha > 0$ , it follows from our Appendix that the negative binomial distribution is the limit law of a pure-death branching process with immigration. But when  $\alpha \in (0, 1)$ , the negative binomial distribution is also compound-geometric, so it is also the limit law of a Markov chain with total disasters, as defined in the next section.

### 4 Compound Poisson-Geometric Distributions

With parameters  $\alpha > 0$ ,  $p \in (0, 1)$ ,  $q := 1 - p$ , a compound Poisson-geometric rv is the sum of  $N$  iid Poisson( $\alpha$ )-distributed rvs, where  $N + 1$  is geometrically distributed with parameter  $p$ . The PLSt and LLt of a compound Poisson-geometric rv are

$$\begin{aligned} \Phi(\lambda) &= \frac{q}{1 - pe^{-\alpha(1-e^{-\lambda})}}, \\ L(\lambda) &:= -\log \Phi(\lambda) = -\log q + \log \left( 1 - pe^{-\alpha(1-e^{-\lambda})} \right). \end{aligned}$$

The mean and variance are

$$\begin{aligned} \mu &= L'(0) = \frac{\alpha p}{q}, \\ \sigma^2 &= -L''(0) = \frac{\alpha p(\alpha + q)}{q^2} = \frac{\alpha + q}{q} \mu > \mu. \end{aligned}$$

Overdispersion holds. Taylor’s law holds with exponent

$$b = \frac{\log \sigma^2}{\log \mu} = 1 + \frac{\log \left( 1 + \frac{\alpha}{q} \right)}{\log \left( \frac{\alpha p}{q} \right)}. \tag{11}$$

The TL exponent  $b$  diverges when  $q = q_c := \alpha / (1 + \alpha)$  or  $\alpha = q / p$ . It vanishes when  $q = q_0$ , defined as the solution in  $(0, 1)$  of

$$f(q) := (1 + \alpha)q^2 - q\alpha(1 - \alpha) - \alpha^2 = 0.$$

Further, because  $f(q_c) = -\alpha^2 / (1 + \alpha) < 0$ , we have  $q_0 > q_c$ . So the graph of  $b$  has two branches for a given fixed  $\alpha$ :



- one is increasing and convex from  $b = 2$  to  $b = \infty$  as  $q$  varies from  $0^+$  to  $q_c^-$ .
- one is increasing from  $b = -\infty$  to  $b = 1$  as  $q$  increases from  $q_c^+$  to 1, passing through 0 as  $q = q_0 > q_c$ . Hence

$$b \in (2, \infty) \text{ if } 0 < q < q_c,$$

$$b < 1 \text{ if } q > q_c.$$

The TL exponent  $b$  is excluded from (1, 2).

If  $\alpha > 0$  is small enough that  $e^{-\alpha} < (1 - \alpha)/\alpha$  (e.g., if  $\alpha < 0.659$ ) and if  $p \geq p_* := \alpha/(1 - e^{-\alpha}) \in (0, 1)$ , then the compound Poisson-geometric rv is SD.

To prove this claim, we observe that, with  $\varphi(z) = e^{-\alpha(1-z)}$

$$\phi(z) = \frac{q}{1 - p\varphi(z)} = e^{-(\log q)[1-c(\varphi(z))]}$$

where  $c(z)$  is the pgf of a logarithmic series distribution. So  $\phi(z)$  is compound Poisson with clusters’ size pgf  $c(\varphi(z))$ .

The probability mass function  $P[X = k] = [z^k]\phi(z)$  of a compound Poisson-geometric rv  $X$  is in principle explicitly given by the Faa di Bruno formula for compositions of two pgfs [18,p. 146]. It involves the ordinary Bell polynomials in the variables  $b_k = [z^k]\varphi(z)$ .

Next, we try to write the pgf  $\phi(z)$  of  $X$  in the form

$$\phi(z) = \frac{q}{1 - p\varphi(z)} = \exp \left\{ -r \int_z^1 \frac{1 - h(z')}{1 - z'} dz' \right\}$$

for some rate  $r > 0$  and pgf  $h(z)$  obeying  $h(0) = 0$ . Forcing  $h(0) = 0$  yields

$$r = \frac{p\varphi\varphi'(0)}{1 - p\varphi(0)} = \frac{p\alpha e^{-\alpha}}{1 - p e^{-\alpha}}.$$

Hence

$$h(z) = 1 - \frac{1}{r}(1 - z) \frac{p\varphi'(z)}{1 - p\varphi(z)} = \frac{1}{r} \frac{r(1 - p\varphi(z)) - p(1 - z)\varphi'(z)}{1 - p\varphi(z)}.$$

Denoting the numerator by  $N(z)$ , a sufficient condition for  $h$  to be a pgf is that

$$[z^k]N(z) \geq 0 \quad \text{for all } k \geq 1,$$

where  $[z^k]N(z)$  denotes the coefficient of  $z^k$  in the power series expansion of  $N(z)$ . But, with  $b_k = e^{-\alpha}\alpha^k/k!$ , we have

$$N(z) = p \sum_{k \geq 1} z^k [(k - r)b_k - (k + 1)b_{k+1}].$$

So  $h$  is a pgf if

$$\frac{b_{k+1}}{b_k} = \frac{\alpha}{k + 1} \leq \frac{p(k - b_1)}{k + 1} \quad \text{for any } k \geq 1$$

or equivalently

$$\alpha \leq p(1 - b_1) = p(1 - e^{-\alpha}) \text{ or } p \geq p_* := \alpha/(1 - e^{-\alpha}).$$

This completes the proof that, under the above condition, the compound Poisson-geometric rv is SD.

A compound Poisson-geometric rv is the limiting rv of a Markov chain with total disasters. To see this, let  $(\beta_n)_{n \geq 1}$  be an iid sequence taking values in  $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ . Let  $\varphi(z) := E(z^\beta)$  be the pgf of the  $\beta$ s. The Markov chain  $X_n$  evolves by:

$$\begin{aligned} X_{n+1} &= X_n + \beta_{n+1} && \text{with probability } p, \\ X_{n+1} &= 0 && \text{with probability } q := 1 - p. \end{aligned}$$

This simple population growth model alternates periods of births of amplitude  $\beta_{n+1}$  with one or more total disasters where population size  $X_n$  is instantaneously reset to 0. We assume without loss of generality that  $X_0 = 0$  and that  $q$  does not depend on  $X_n = x$ . We define  $p * X$  to equal the product of  $X$  times an independent Bernoulli rv  $B(p)$  with success parameter  $p$ . (This product is not to be confused with Bernoulli thinning.) Because  $X := X_\infty$  solves the distributional equation  $X \stackrel{d}{=} B(p)(X' + \beta) =: p * (X' + \beta)$ , where  $(B(p), X' \stackrel{d}{=} X, \beta)$  are mutually independent and  $B(p)$  is a Bernoulli rv with success parameter  $p$ , the above Markov chain  $\{X_n \mid n = 0, 1, 2, \dots\}$  with total disasters is clearly ergodic. The limiting rv  $X := X_\infty$  exists and it has the compound geometric pgf

$$\phi(z) = q / (1 - p\varphi(z))$$

(shifted to the left by one unit). In our example,  $\beta$  is Poisson-distributed with pgf  $\varphi(z) = e^{-\alpha(1-z)}$ .

### 5 Compound Geometric-Poisson Distributions

With parameters  $\alpha > 0$ ,  $p \in (0, 1)$ ,  $q := 1 - p$ , a compound geometric-Poisson (or Pólya-Aeppli) rv is the sum of a Poisson( $\alpha$ )-distributed number of geometric rvs with parameter  $p$ . The PLSt and LLt are

$$\begin{aligned} \Phi(\lambda) &= e^{-\alpha \left(1 - \frac{qe^{-\lambda}}{1 - pe^{-\lambda}}\right)} = e^{-\alpha \frac{1 - e^{-\lambda}}{1 - pe^{-\lambda}}}, \\ L(\lambda) &:= -\log \Phi(\lambda) = \alpha \frac{1 - e^{-\lambda}}{1 - pe^{-\lambda}}. \end{aligned}$$

Hence the mean and variance are

$$\begin{aligned} \mu &= L'(0) = \frac{\alpha}{q} > 0, \\ \sigma^2 &= -L''(0) = \frac{\alpha(1+p)}{q^2} = \frac{1+p}{q} \mu > \mu. \end{aligned}$$

Its probability mass function is given by

$$\begin{aligned} P[X = k] &= e^{-\alpha} \sum_{l=1}^k \binom{k-1}{l-1} \frac{\alpha^l}{l!} p^{k-l} q^l, \text{ if } k \geq 1, \\ &= e^{-\alpha} \text{ if } k = 0. \end{aligned}$$

Overdispersion holds. Taylor’s law holds with exponent

$$b = \frac{\log \sigma^2}{\log \mu} = 1 + \frac{\log \left(\frac{1+p}{q}\right)}{\log \left(\frac{\alpha}{q}\right)}. \tag{12}$$

If  $\alpha \geq 2$ , the graph of  $b$  has only one decreasing branch as  $q$  varies from 0 to 1, with  $b \rightarrow 2^-$  as  $q \rightarrow 0^+$  and  $b = 1$  at  $q = 1$ . Hence  $b \in [1, 2]$ .

If  $1 < \alpha < 2$ , the graph of  $b$  increases from  $b = 2$  to a maximum less than 3 and then drops to 1 as  $q$  increases from 0 to 1.

If  $\alpha = 1$ , then  $b = \log(1 + p) = \log(2 - q)$ , so  $b$  falls from  $\log 2$  to 0 as  $q$  increases from 0 to 1.

If  $0 < \alpha < 1$ ,  $b$  diverges when  $q = q_c := \alpha$ , so  $b$  has two branches:

- one is increasing and convex from  $b = 2^+$  to  $b = +\infty$  as  $q$  increases from  $0^+$  to  $q_c^-$ .
- one is increasing and concave from  $b = -\infty$  to  $b = 1$  as  $q$  increases from  $q_c^+$  to  $1^-$ .

This branch passes through  $b = 0$  when  $q = q_0 := (\sqrt{\alpha^2 + 8\alpha - \alpha})/2 > q_c$ . Hence the full range of  $b$  is covered:

$$b \in (2, +\infty) \text{ if } 0 < q < q_c,$$

$$b \in (-\infty, 1) \text{ if } q > q_c.$$

The pgf  $\phi(z)$  of a geometric-Poisson or Pólya-Aeppli rv is

$$\phi(z) = \exp \left\{ -\alpha \frac{1-z}{1-pz} \right\} = \exp \left\{ -r \int_z^1 \frac{1-h(z')}{1-z'} dz' \right\}$$

for some rate  $r > 0$  and pgf  $h(z)$  obeying  $h(0) = 0$ , only if  $p > 1/2$ . The necessary condition  $p > 1/2$  arises because  $h(0) = 0$  yields  $r = \alpha q$  and

$$h(z) = z \frac{1 - 2p + p^2 z}{(1 - pz)^2},$$

which is absolutely monotone only if  $p > 1/2$ , and then  $h(z)$  satisfies  $[z^n]h(z) \geq 0$  for all  $n \geq 1$ .

### 6 Gamma Distributions

A gamma rv  $X$  with shape parameter  $\alpha > 0$  and scale parameter  $\beta > 0$  has probability law

$$P(X \in dx) = \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha} dx, \quad x \in (0, \infty).$$

The PLSt and LLt are

$$\Phi(\lambda) = \left( \frac{1}{1 + \lambda\beta} \right)^\alpha,$$

$$L(\lambda) = -\log \Phi(\lambda) = \alpha \log(1 + \lambda\beta).$$

The mean and variance are

$$\mu = L'(0) = \alpha\beta > 0,$$

$$\sigma^2 = -L''(0) = \alpha\beta^2 = \frac{1}{\alpha}\mu^2 = \beta\mu.$$

Overdispersion holds if and only if  $\beta > 1$ .

As a reviewer pointed out, a gamma rv can satisfy TL in multiple ways. For example, if we fix  $b = 2$ , then  $\sigma^2/\mu^2 = 1/\alpha = A$  and  $a = -\log \alpha$ . If we fix  $b = 1$ , then  $\sigma^2/\mu = \beta = A$

and  $a = \log \beta$ . If we fix  $b = 3/2$ , then  $\sigma^2/\mu^{3/2} = (\beta/\alpha)^{1/2}$  and  $a = (1/2) \log(\beta/\alpha)$ . Here we fix  $a = 0$ ,  $A = 1$  so that TL holds with exponent

$$b = \frac{\log \sigma^2}{\log \mu} = 1 + \frac{\log \beta}{\log(\alpha\beta)}. \tag{13}$$

If  $\beta = 1$ , then  $b = 1$  constant (equidispersion, or variance equal to the mean).

If  $\beta \neq 1$ , the graph of  $b$  versus  $\alpha$  shows a singularity at  $\alpha_c := 1/\beta$ , with the full range of  $b$  covered.

If  $\beta < 1$ , the graph of  $b$  versus  $\alpha$  has two increasing branches. We have  $b \rightarrow 1^+$  if  $\alpha \rightarrow 0^+$ ,  $b \rightarrow +\infty$  if  $\alpha \rightarrow \alpha_c^-$  and  $b \rightarrow -\infty$  if  $\alpha \rightarrow \alpha_c^+$ ,  $b \rightarrow 1^-$  if  $\alpha \rightarrow \infty$ .

If  $\beta > 1$ , the graph of  $b$  is a symmetric image of the previous one with respect to the horizontal line  $b = 0$ , so with two decreasing branches.

If  $\alpha = 1$ , giving an exponential distribution, then  $b = 2$  is a fixed point.

The gamma rv is SD because [30,p. 4, Theorem 2.9]

$$\Phi(\lambda) = \exp \left\{ \int_0^\lambda \frac{\log \Phi_0(\lambda')}{\lambda'} d\lambda' \right\}$$

for some PLSt  $\Phi_0(\lambda)$  of an infinitely divisible rv, which is here found to be

$$\Phi_0(\lambda) = e^{\alpha\beta\lambda/(1+\lambda\beta)} = e^{-\alpha(1-1/(1+\lambda\beta))}.$$

This  $\Phi_0(\lambda)$  is the PLSt of a compound Poisson( $\alpha$ ) rv with Exp( $\beta$ ) distribution for the random size  $\Delta$  of its clusters. Consider the Ornstein–Uhlenbeck process

$$dX_t = -X_t dt + d\mathcal{L}_t, \quad X_0 = 0,$$

driven by the Lévy-process  $\mathcal{L}_t$  (here a rate- $\alpha$  compound-Poisson exponential process) for which

$$E e^{-\lambda\mathcal{L}_t} = \Phi_0(\lambda)^t, \quad t \geq 0.$$

With  $\phi_\Delta(\lambda) = E e^{-\lambda\Delta} = (1 + \lambda\beta)^{-1}$ ,

$$\begin{aligned} \Phi_t(\lambda) &:= E \left( e^{-\lambda X_t} \mid X_0 = 0 \right) = \exp \left\{ -\alpha \int_0^t (1 - \phi_\Delta(\lambda e^{-s})) ds \right\} = \left( \frac{1 + \lambda\beta}{1 + \lambda\beta e^{-t}} \right)^{-\alpha} \\ &\rightarrow (1 + \lambda\beta)^{-\alpha} = E \left( e^{-\lambda X_\infty} \mid X_0 = 0 \right) = \Phi(\lambda) \text{ as } t \rightarrow \infty. \end{aligned}$$

The function  $\Phi_t(\lambda)$  is the PLSt of some rv  $X_t$  with an atom at 0 with probability mass  $e^{-\alpha t}$ .

### 7 Scaling

In each example above, scaling the log-Laplace transform according to

$$L(\lambda) \rightarrow L_1(\lambda) = \frac{1}{\sigma_1^2} L(\sigma_1^2 \lambda) \tag{14}$$

defines the law of a new rv. Indeed, the scaled function  $L_1(\lambda)$  defined from the LLt  $L(\lambda)$  is itself the LLt of some rv if and only  $L(\lambda)$  is the LLt of some infinitely-divisible rv (which is true in our examples here).

Under this scaling (14), the mean  $\mu = L_1'(0)$  remains invariant while the variance is rescaled from  $-L''(0) = \sigma^2$  to  $-L_1''(0) = \sigma_1^2 \sigma^2$ . (For instance, a  $\Gamma(\alpha, \beta)$  rv transforms to a  $\Gamma(\alpha/A, \beta A)$  rv with  $A = \sigma_1^2$ .) Then TL transforms according to

$$\sigma^2 = \mu^b \rightarrow \sigma^2 = \sigma_1^2 \mu^b$$

where  $\sigma_1^2$  (previously named  $A$ ) is the variance of the new scaled rv when its mean is 1. The log-linear version of TL now includes a non-zero constant term  $a = \log \sigma_1^2$ :

$$\log \sigma^2 = b \log \mu \rightarrow \log \sigma^2 = a + b \log \mu.$$

If  $X$  has support  $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ , then under the scaling (14) its support becomes  $e^a \mathbb{N}_0 := \{0, e^a, 2e^a, \dots\}$ . Scaling in the discrete case changes the original 'counting' support  $\mathbb{N}_0$  to  $e^a \mathbb{N}_0$ , emphasizing that  $A = e^a$  plays the role of a scale parameter of the underlying distribution. In this discrete dispersion model,  $A = e^a$  could be interpreted as the individual mass of each individual constituting the population [25].

If the support of  $X$  is the whole real line or the nonnegative half line, the above rescaling by  $e^a$  does not change the support.

## 8 Conclusion and Open Questions

Empirical observations that sample means and sample variances are approximately consistent with TL cannot specify the underlying distribution, although they can reject some possibilities, as when the best estimates of  $a \neq 0$  or  $b \neq 1$  reject a Poisson distribution. As [15] observed, for every  $\mu > 0$  and any rv  $Y$  with mean 0 and variance 1, the rv  $X = \mu + \mu^{b/2} Y$  obeys TL with exponent  $b$  and intercept  $a = 0$ .  $X$  can then be scaled following (14) to include the affine term  $a$ , resulting in a model obeying TL with any  $a$  and  $b$ .

For each of the infinitely divisible two-parameter models in this note, we showed that the TL exponent  $b$  depends on the parameters in very specific ways. We showed that the admissible range of  $b$  depends on the distribution, and in some cases on the parameters of the distribution, in ways that could help identify the underlying law from data approximately obeying TL, or at least could exclude some possible underlying laws.

The expression  $b = (2 - \alpha)/(1 - \alpha)$  for the TL exponent in (6) arose in at least three different prior examples of TL. First, TL holds asymptotically (as sample size increases toward infinity) with exponent  $b = (2 - \alpha)/(1 - \alpha)$  for the sample variance and sample mean of samples from nonnegative stable distributions with tail index  $\alpha \in (0, 1)$  [12, p. 663, Proposition 2]. Such distributions have infinite mean. Second, more generally, TL holds asymptotically with exponent  $b = (2 - \alpha)/(1 - \alpha)$  for the sample variance and sample mean of samples of increasing size from nonnegative distributions with regularly varying upper tails and tail index  $\alpha \in (0, 1)$  [17, p. 6, their Eq. (3.2)]. Again, such distributions have infinite mean. Third, TL holds asymptotically with exponent  $b = (2 - \alpha)/(1 - \alpha)$  for samples of increasing size from nonnegative distributions with regularly varying upper tails and tail index  $\alpha \in (0, 1)$  when the sample variance is replaced by the sample upper semivariance [13, p. 4, Theorem 2, their Eq. 19].

By contrast, for samples of increasing size from nonnegative distributions with regularly varying upper tails and tail index  $\alpha \in (0, 1)$ , TL holds asymptotically, but with different formulas for the exponent  $b$ , when the sample variance is replaced by the sample lower semivariance ( $b = 2$ , regardless of  $\alpha \in (0, 1)$ ), the sample lower local semivariance ( $b = 2$ ,

regardless of  $\alpha \in (0, 1)$ ), or the sample local upper semivariance ( $b = (2 - \alpha^2)/(1 - \alpha)$ ) [13].

Can four appearances of (6), one here and three earlier, be coincidences? Or is some underlying process or mechanism common to all these different appearances?

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**Data Availability** There are no data associated with this paper.

## Declarations

**Conflict of interest** The authors have no conflicts of interest associated with this paper.

## Appendix

Here we briefly sketch that if  $X$  is SD, it is a weak limit of a pure-death branching process with immigration (if  $\mathbb{N}_0$ -valued), or a weak limit of a continuous-time Lévy-driven Ornstein-Uhlenbeck process (if continuous). The analysis concerns the rvs  $X$  obeying TL with  $a = 0$ . From the scaling transform (14) introducing  $a = \log \sigma_1^2$ , the modifications for  $a \neq 0$  could be readily obtained. In both discrete and continuous cases, a population that is randomly annihilated is randomly regenerated by the recurrent arrivals of random quantities of immigrants, yielding a stationary or invariant distribution of population size.

### Discrete Self-decomposable rvs and Pure-Death Branching Processes with Immigration in Continuous Time

van Harn et al. [33] construct a regenerative process in continuous time that produces discrete SD distributions in the long run. Consider a continuous-time homogeneous compound Poisson process  $P_r(t)$ ,  $t \geq 0$ ,  $P_r(0) = 0$ , having rate  $r > 0$ , with pgf

$$E_{P_r(0)=0}(z^{P_r(t)}) = \exp\{-rt(1 - h(z))\}, \tag{15}$$

where  $h(z)$ , with  $h(0) = 0$ , is the pgf of the sizes of the clones or immigrant clusters arriving at the jump times of  $P_r(t)$ . Let

$$\varphi_t(z) = 1 - e^{-t}(1 - z) \tag{16}$$

be the pgf of a pure-death branching process started with one particle at  $t = 0$ . (More general subcritical branching processes could be considered.) This expression of  $\varphi_t(z)$  is easily seen to solve  $\dot{\varphi}_t(z) = f(\varphi_t(z)) = 1 - \varphi_t(z)$ ,  $\varphi_0(z) = z$ , as is usual for a pure-death continuous-time Bellman-Harris branching process [21] with affine branching mechanism  $f(z) = r_d(1 - z)$  with fixed death rate  $r_d = 1$ . The distribution function of the lifetime of the initial particle is  $1 - e^{-t}$ . Let  $X_t$  with  $X_0 = 0$  be a random process counting the current size of some population for which a random number of individuals (determined by  $h(z)$ ) immigrate at the jump times of  $P_r(t)$ . Each newly arrived individual is independently and

immediately subject to the pure death process above. We have

$$\phi_t(z) := E(z^{X_t}) = \exp \left\{ -r \int_0^t [1 - h(\phi_s(z))] ds \right\}, \quad \phi_0(z) = 1, \tag{17}$$

with  $\phi_t(0) = P(X_t = 0) = \exp\{-r \int_0^t (1 - h(1 - e^{-s})) ds\}$ , the probability that the population is extinct at  $t$ . As  $t \rightarrow \infty$ ,

$$\begin{aligned} \phi_t(z) &\rightarrow \phi_\infty(z) = \exp \left\{ -r \int_0^\infty [1 - h(1 - e^{-s}(1 - z))] ds \right\} \\ &= \exp \left\{ -r \int_z^1 \frac{1 - h(u)}{1 - u} du \right\}. \end{aligned} \tag{18}$$

So  $X := X_\infty$ , as the limiting population size of this pure-death process with immigration, is a SD rv [33]. Define the rv  $X_c$  implicitly by requiring that  $X \stackrel{d}{=} cX' + X_c$ , where  $X'$  is an iid copy of  $X$  and  $0 < c < 1$ . Then

$$\phi_{X_c}(z) = \frac{\phi_\infty(z)}{\phi_\infty(1 - c(1 - z))} = \exp \left\{ -r \int_z^{1-c(1-z)} \frac{1 - h(u)}{1 - u} du \right\}$$

is a pgf. In such models typically, a decaying subcritical branching population is regenerated by a random number of incoming immigrants at random Poissonian times.

### Continuous Self-decomposable rvs and Ornstein–Uhlenbeck Process in Continuous-Time

When  $X$  is continuous and SD,  $X$  is the limiting distribution of population size as  $t \rightarrow \infty$  of some Ornstein-Uhlenbeck process  $X_t$ :

$$dX_t = -X_t dt + d\mathcal{L}_t, \quad X_0 = 0,$$

driven by the Lévy process  $\mathcal{L}_t$  for which

$$Ee^{-\lambda\mathcal{L}_t} = \Phi_0(\lambda)^t, \quad t \geq 0,$$

where  $\Phi_0(\lambda)$  is the PLSt of an infinitely divisible rv appearing in the representation (7) of  $\Phi(\lambda) = Ee^{-\lambda X}$ . See [26].

We now show that for a TweBLE rv with  $\alpha \in (-\infty, 0)$ , there is no  $L_0(\lambda) = -\log \Phi_0(\lambda)$  such that  $L'_0(\lambda)$  is completely monotone on  $(0, \infty)$ . This result means that the TweBLE rv for  $\alpha \in (-\infty, 0)$  is not SD, just infinitely divisible. Indeed, with  $L(\lambda) = -\log \Phi(\lambda)$ ,

$$\begin{aligned} L_0(\lambda) &= \lambda L'(\lambda) = (1 - \alpha)^{1-\alpha} \lambda (\theta + \lambda)^{\alpha-1}, \\ L'_0(\lambda) &= (1 - \alpha)^{1-\alpha} (\theta + \lambda)^{\alpha-2} [\theta + \lambda \alpha], \end{aligned}$$

with  $L'_0(\lambda) > 0$  only if  $\lambda > \lambda_c = -\theta/\alpha > 0$ , so not in the full range  $\lambda \in (0, \infty)$ . So  $\Phi_0(\lambda)$  is not completely monotone on  $(0, \infty)$ , and is therefore not an infinitely divisible PLSt. This Poisson-gamma regime for which the limiting distribution is a Poisson sum  $P$  of iid gamma-distributed clusters of size  $\Delta$  was studied by [19, p. 17, Sect. 3.3.2], who underline what they call its “impact inhomogeneity”:  $E(\Delta) = C(\alpha) \cdot E(P)^{-1/\alpha}$  for some constant  $C(\alpha) > 0$ .

By contrast, if  $\alpha \in (0, 1)$ , then  $L_0(\lambda)$  may be written as

$$L_0(\lambda) = \int_0^{\infty} (1 - e^{-\lambda x}) \pi(x) dx$$

where

$$\pi(x) dx = \frac{1}{\Gamma(1-\alpha)} x^{-(\alpha+1)} (\alpha + \theta x) e^{-\theta x} dx$$

is a tempered Lévy measure integrating  $1 \wedge x$ . The driving process  $\mathcal{L}_t$  of  $X_t$ , with PLSt  $Ee^{-\lambda \mathcal{L}_t} = e^{-tL_0(\lambda)}$ , is a subordinator and  $X = X_{\infty}$  is a SD TweBLE rv obtained as the limiting distribution of the corresponding Ornstein-Uhlenbeck process. One could extend this construction to other SD subordinated Lévy families, such as those in [1, 6, 10, 28, 29].

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