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Cauchy, normal and correlations versus heavy tails[☆]

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ABSTRACT

A surprising result of Pillai and Meng (2016) showed that a transformation $\sum_{j=1}^{n} w_j X_j / Y_j$ of two iid centered normal random vectors, $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$, $n > 1$, for any weights $0 \le w_j \le 1$, $j = 1, \ldots, n$, $\sum_{j=1}^{n} w_j = 1$, has a Cauchy distribution regardless of any correlations within the normal vectors. The correlations appear to lose out in the competition with the heavy tails. To clarify how extensive this phenomenon is, we analyze two other transformations of two iid centered normal random vectors. These transformations are similar in spirit to the transformation considered by Pillai and Meng (2016). One transformation involves absolute values: $\sum_{j=1}^{n} w_j X_j / |Y_j|$. The second involves randomly stopped Brownian motions: $\sum_{j=1}^{n} w_j X_j \left(Y_j^{-2}\right)$, where $\left\{\left(X_1(t), \ldots, X_n(t)\right),\ t \ge 0\right\}$, $n > 1$, is a Brownian motion with positive variances; $(Y_1, \ldots, Y_n)$ is a centered normal random vector with the same law as $(X_1(1), \ldots, X_n(1))$ and independent of it; and $X(Y^{-2})$ is the value of the Brownian motion $X(t)$ evaluated at the random time $t = Y^{-2}$. All three transformations result in a Cauchy distribution if the covariance matrix of the normal components is diagonal, or if all the correlations implied by the covariance matrix equal 1. However, while the transformation that Pillai and Meng considered produces a Cauchy distribution regardless of the normal covariance matrix, the transformations we consider here do not always produce a Cauchy distribution. The correlations between jointly normal random variables are not always overwhelmed by the heaviness of the marginal tails. The mysteries of the connections between normal and Cauchy laws remain to be understood.

1. Introduction

The normal distribution and the Cauchy distribution are among the most fundamental probability distributions with numerous applications in many areas where uncertainty is present. They both arise as limits of suitably normalized sums of independent copies of random quantities. They both have nearly magical invariance properties. They contrast strongly in that the normal distribution has finite moments of all orders (even all exponential moments are finite), while the

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Cauchy distribution does not even have a finite mean. It is well known that normal and Cauchy random variables are connected in simple ways. These connections sometimes appear to have mysterious properties as well. This note aims to shed additional light on some of these mysteries.

By definition, the centered normal distribution with standard deviation $\sigma > 0$ has the density

$$\phi(x; \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\{-x^2/(2\sigma^2)\}, \ x \in \mathbb{R},$$

while the Cauchy distribution with scale $\sigma > 0$ has the density

$$f(x; \sigma) = \frac{\sigma}{\pi(x^2 + \sigma^2)}, \ x \in \mathbb{R}. \tag{1.1}$$

For both distributions, the case $\sigma = 1$ is called standard. The invariance properties of these distributions say that, if $X_1, \dots, X_n$ are iid copies of a centered normal X and $\overset{d}{=}$ denotes equality in distribution, then

$$\sum_{j=1}^{n} w_j X_j \overset{d}{=} X \tag{1.2}$$

if $\sum_{j=1}^{n} w_j^2 = 1$, while if $X_1, \dots, X_n$ are iid copies of a Cauchy X, then (1.2) holds if $\sum_{j=1}^{n} |w_j| = 1$.

Versions of (1.2) may hold when the $X_1, \dots, X_n$ are not independent. For example, recall that $X_1, \dots, X_n$ are jointly Cauchy if there is a finite symmetric measure Γ (the spectral measure) on the unit sphere S^n such that

$$E \exp\left\{ i \sum_{j=1}^{n} \theta_j X_j \right\} = \exp\left\{ - \int_{S^n} \left| \sum_{j=1}^{d} \theta_j s_j \right| \Gamma(ds_1, \dots, ds_n) \right\}. \tag{1.3}$$

Because $(X_1, \dots, X_n)$ has a symmetric distribution, the characteristic function is real-valued on the right side of (1.3). Then $X_1, \dots, X_n$ have identical marginals if $\int |s_j| \, \Gamma(d\mathbf{s})$ is the same for all j; see Samorodnitsky and Taqqu (1994). In particular, if the spectral measure Γ is concentrated on the positive and negative quadrants of the unit sphere and $X_1, \dots, X_n$ have the same marginals, then (1.2) still holds as long as all (w_j) are also of the same sign. In this scenario, $X_1, \dots, X_n$ are generally dependent since their independence requires Γ to be concentrated at the points where the unit sphere intersects the axes. In particular, if Γ is concentrated at the points $\pm(1, \dots, 1)/n^{1/2}$, then $X_1 = \cdots = X_n$ a.s. This special case shows that a convex linear combination of iid Cauchy random variables has the same law as a convex linear combination of almost surely equal Cauchy random variables!

2. The Pillai and Meng result

An elementary calculation shows that, if X, Y are iid centered normal and W is a standard Cauchy, then

$$X/Y \overset{d}{=} W. \tag{2.1}$$

If X and Y are bivariate normal, then the probability density function of the ratio X/Y is derived in Pham-Gia et al. (2006) and earlier sources they cite. The ratio has a Cauchy distribution if and only if X and Y are centered and uncorrelated. For the remainder of this note, we will take X and Y to be centered and uncorrelated.

Pillai and Meng (2016) proved an amazing extension of (2.1). Let $(X_1, \dots, X_n)$ and $(Y_1, \dots, Y_n)$, $n > 1$, be iid multivariate centered normal random vectors with positive variances and a covariance matrix Σ. Then for any weights $0 \le w_j \le 1, j = 1, \dots, n, \ \sum_{j=1}^{n} w_j = 1$,

$$\sum_{j=1}^{n} w_j \frac{X_j}{Y_j} \overset{d}{=} W, \tag{2.2}$$

where W is a standard Cauchy. Earlier, Drton and Xiao (2016) proved (2.2) when $n = 2$.

What is amazing in (2.2)? Each ratio X_j/Y_j in the sum has the standard Cauchy distribution by (2.1). If the covariance matrix Σ is diagonal, then these Cauchy random variables are independent, and (2.2) is simply the invariance property (1.2). If, on the other hand, all the correlations implied by the covariance matrix Σ equal 1, then all terms in (2.2) are equal almost surely, and (2.2) is equivalent to the above-mentioned fact that a convex linear combination of equal Cauchy random variables has the same law as a convex linear combination of iid Cauchy random variables. The amazing part is that (2.2) holds for all other covariance matrices Σ.

The only comparable property of non-independent Cauchy random variables that appears to be known at this point is the property of jointly Cauchy random variables with the spectral measure Γ concentrated on the positive and negative quadrants of the unit sphere. It is straightforward to show that the terms in (2.2) are jointly Cauchy if and only if the normal random vectors consist of independent blocks of equal normal random variables.

Therefore, it is a special feature of the multivariate normal distribution that the ratios of the components of iid centered normal random vectors have the invariance property (1.2), regardless of the correlations among the normal components.

Pillai and Meng (2016) speculate that "the dependence among them" (i.e., among the ratios of centered, dependent normal random variables) "can be overwhelmed by the heaviness of their marginal tails" (of the ratios) "in determining the stochastic behavior of their linear combinations". This speculation continues to generate interest and new results. For example, Cohen et al. (2020) give another example where heavy tails overwhelm correlations among normal random variables.

Here we shall show, by two examples, that the correlations between jointly normal random variables are not always overwhelmed by the heaviness of the marginal tails. In the two subsequent sections, we analyze two other natural transformations, in the spirit of Pillai and Meng (2016), from the normal world to the Cauchy world.

3. Transformation 1: Absolute values

The symmetry of a centered normal random variable and (2.1) imply that, if X, Y are iid centered normals and W is a standard Cauchy, then

$$X/|Y| \stackrel{d}{=} W. \tag{3.1}$$

Following Pillai and Meng (2016), let $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$, $n > 1$, be iid multivariate centered normal random vectors with positive variances and a covariance matrix Σ. Is it true that, for any weights $0 \le w_j \le 1$, $j = 1, \ldots, n$, $\sum_{j=1}^n w_j = 1$,

$$\sum_{j=1}^{n} w_j \frac{X_j}{|Y_j|} \stackrel{d}{=} W, \tag{3.2}$$

where W is a standard Cauchy? As was the case with (2.2), the claim (3.2) holds if the covariance matrix Σ is a diagonal matrix, and also when all the correlations implied by the covariance matrix Σ equal 1. However, we will show that (3.2) is not true for some covariance matrices Σ. The argument uses a simple property of Cauchy densities that follows immediately from (1.1):

$$f(x;\sigma) = \frac{f(0;\sigma)}{\pi^2 f^2(0;\sigma)x^2 + 1}, \quad x \in \mathbb{R}. \tag{3.3}$$

It is enough to show that (3.2) does not, in general, hold when $n = 2$. A non-singular covariance matrix Σ with equal variances can be scaled so that its inverse matrix is

$$\Sigma^{-1} = \begin{bmatrix} 1 & \theta \\ \theta & 1 \end{bmatrix} \quad \text{for } -1 < \theta < 1. \tag{3.4}$$

Let $R_i = X_i/|Y_i|$, $i = 1, 2$. An elementary calculation for the joint density of (R_1, R_2) gives, in the parametrization (3.4),

$$\begin{aligned}
& f_{R_1,R_2}(r_1, r_2) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(1-\theta^2)|y_1y_2|}{(2\pi)^2} \cdot \\
&\qquad \exp\left\{-\frac{1}{2}(y_1^2r_1^2 + 2\theta|y_1y_2|r_1r_2 + y_2^2r_2^2) - \frac{1}{2}(y_1^2 + 2\theta y_1y_2 + y_2^2)\right\} dy_1dy_2 \\
&= 2\int_0^{\infty}\int_0^{\infty} \frac{(1-\theta^2)xy_2^3}{(2\pi)^2} \exp\left\{-\frac{1}{2}\left(x^2y_2^2(r_1^2+1) + 2\theta xy_2^2(r_1r_2+1) + y_2^2(r_2^2+1)\right)\right\} dxdy_2 \\
&+ 2\int_0^{\infty}\int_0^{\infty} \frac{(1-\theta^2)xy_2^3}{(2\pi)^2} \exp\left\{-\frac{1}{2}\left(x^2y_2^2(r_1^2+1) + 2\theta xy_2^2(r_1r_2-1) + y_2^2(r_2^2+1)\right)\right\} dxdy_2 \\
&= \int_0^{\infty} \frac{(1-\theta^2)x}{\pi^2\left(x^2(r_1^2+1) + 2\theta x(r_1r_2+1) + (r_2^2+1)\right)^2} dx \\
&\quad + \int_0^{\infty} \frac{(1-\theta^2)x}{\pi^2\left(x^2(r_1^2+1) + 2\theta x(r_1r_2-1) + (r_2^2+1)\right)^2} dx.
\end{aligned}$$

Define the convex combination of R_1 and R_2 to be $V := w_1R_1 + w_2R_2$, where $w_1 \in [0, 1]$ and $w_2 = 1 - w_1$. Then the density g_V of V is given by

$$\begin{aligned}
g_V(v) = &\int_{-\infty}^{\infty}\int_0^{\infty} \frac{(1-\theta^2)w_2^3x}{\pi^2\left(w_2^2x^2(r^2+1) + 2\theta w_2x(r(v-w_1r)+w_2) + ((v-w_1r)^2+w_2^2)\right)^2} dxdr \\
&+ \int_{-\infty}^{\infty}\int_0^{\infty} \frac{(1-\theta^2)w_2^3x}{\pi^2\left(w_2^2x^2(r^2+1) + 2\theta w_2x(r(v-w_1r)-w_2) + ((v-w_1r)^2+w_2^2)\right)^2} dxdr
\end{aligned} \tag{3.5}$$

$$= \int_0^\infty \frac{(1-\theta^2)(w_2^2x^2 - 2\theta w_1 w_2 x + w_1^2)x}{2\pi\left((w_2^2x^2 - 2\theta w_1 w_2 x + w_1^2)(x^2+2\theta x+1) + (1-\theta^2)v^2x^2\right)^{3/2}}dx$$
$$+ \int_0^\infty \frac{(1-\theta^2)(w_2^2x^2 - 2\theta w_1 w_2 x + w_1^2)x}{2\pi\left((w_2^2x^2 - 2\theta w_1 w_2 x + w_1^2)(x^2-2\theta x+1) + (1-\theta^2)v^2x^2\right)^{3/2}}dx.$$

For $w_1 \in (0, 1)$ fixed, we will prove that

$$\lim_{v\to\infty} v^2 g_V(v) = 1/\pi, \tag{3.6}$$

for all $\theta \in (0, 1)$. Assuming that (3.6) is true, if g_V were a Cauchy density for any $\theta \in (0, 1)$ it would follow from (3.3) that

$$g_V(0) = 1/\pi \text{ for any } \theta \in (0, 1).$$

We will show that, in fact,

$$g_V(0) > 1/\pi \tag{3.7}$$

for all $\theta \in (0, \epsilon)$ for some $\epsilon > 0$. This precludes the possibility that g_V is a Cauchy density for all such θ.

To establish (3.7), it suffices to show that for any $w_1 \in (0, 1)$, the derivative of the function $g_V(0)$ with respect to θ is positive at $\theta = 0$. Indeed, by (3.5)

$$g_V(0) = \int_0^\infty \frac{(1-\theta^2)(w_2^2x^2 - 2\theta w_1 w_2 x + w_1^2)x}{2\pi\left(w_2^2x^2 - 2\theta w_1 w_2 x + w_1^2\right)^{3/2}\left(x^2+2\theta x+1\right)^{3/2}}dx$$
$$+ \int_0^\infty \frac{(1-\theta^2)(w_2^2x^2 - 2\theta w_1 w_2 x + w_1^2)x}{2\pi\left(w_2^2x^2 - 2\theta w_1 w_2 x + w_1^2\right)^{3/2}\left(x^2-2\theta x+1\right)^{3/2}}dx,$$

It is elementary that one may differentiate with respect to θ inside the integrals to obtain

$$\left.\frac{\partial g_V(0)}{\partial\theta}\right|_{\theta=0} = \int_0^\infty \frac{w_1 w_2 x^2}{\pi(w_1^2 + w_2^2x^2)^{3/2}(x^2+1)^{3/2}}\,dx > 0,$$

as promised. We conclude that (3.7) holds and, hence (3.2) cannot be valid for θ in a neighborhood of zero.

It remains to prove (3.6). We use the last equality of (3.5) and write in the obvious notation

$$v^2 g_V(v) = v^2\left[\int_1^\infty + \int_1^\infty\right] + v^2\left[\int_0^1 + \int_0^1\right] = I_1(v) + I_2(v).$$

We have, as $v \to \infty$,

$$I_1(v) \sim (1-\theta^2)w_2^2 \int_1^\infty \frac{x^3v^2}{\pi\left(w_2^2x^4 + (1-\theta^2)v^2x^2\right)^{3/2}}\,dx$$
$$= w_2 \int_{w_2/(v(1-\theta^2)^{1/2})}^\infty \frac{1}{\pi(x^2+1)^{3/2}}\,dx \to w_2\int_0^\infty \frac{1}{\pi(x^2+1)^{3/2}}\,dx = w_2/\pi.$$

Since I_2 can be reduced to I_1 by a variable change $x \to 1/x$ and switching w_1 and w_2, this means that $I_2(v) \to w_1/\pi$ as $v \to \infty$. Because $w_1 + w_2 = 1$, this proves (3.6).

In summary, it is not true that, for any choice of (w_1, w_2) with $0 < w_1 < 1,\ w_1 + w_2 = 1$, the sum on the left side of (3.2) has a Cauchy distribution for all θ in an arbitrarily small neighborhood of the origin.

4. Transformation 2: Randomly stopped Brownian motions

Recall that an $\mathbb{R}^n$-valued Brownian motion $\left\{\left(X_1(t), \ldots, X_n(t)\right),\ t \geq 0\right\}$ is a continuous centered Gaussian process that starts at the origin at time zero and has stationary and independent increments. The law of such a process is completely determined by the law of its values at time 1, $\left(X_1(1), \ldots, X_n(1)\right)$. The covariance function of a Brownian motion is

$$\operatorname{cov}(X_i(s), X_j(t)) = \min(s, t)\operatorname{cov}(X_i(1), X_j(1)),\ \ s, t \geq 0,\ \ i, j = 1, \ldots, n. \tag{4.1}$$

A Brownian motion is self-similar in the sense that, for any $c > 0$,

$$\left\{\left(X_1(ct), \ldots, X_n(ct)\right),\ t \geq 0\right\} \stackrel{d}{=} \left\{\left(c^{1/2}X_1(t), \ldots, c^{1/2}X_n(t)\right),\ t \geq 0\right\}. \tag{4.2}$$

If $\left\{X(t),\ t \geq 0\right\}$ is a one-dimensional Brownian motion, Y is an independent copy of $X(1)$ and $X(Y^{-2})$ is the value of the Brownian motion $X(t)$ evaluated at the random time $t = Y^{-2}$, then (4.2) implies that

$$X\left(Y^{-2}\right) \stackrel{d}{=} X(1)/|Y| \stackrel{d}{=} X(1)/Y \stackrel{d}{=} W \tag{4.3}$$

by (2.1), where W is a standard Cauchy.

Again following Pillai and Meng (2016), let $\{(X_1(t), \dots, X_n(t)),\ t \geq 0\},\ n > 1$, be a Brownian motion with positive variances and $(Y_1, \dots, Y_n)$ a centered normal random vector with the same law as $(X_1(1), \dots, X_n(1))$ and independent of it. Let their common covariance matrix be Σ. Is it true that for any weights $0 \leq w_j \leq 1, j = 1, \dots, n, \sum_{j=1}^n w_j = 1$, and any Σ,

$$\sum_{j=1}^{n} w_j X_j\big(Y_j^{-2}\big) \overset{d}{=} W, \tag{4.4}$$

where W is a standard Cauchy? If Σ is diagonal, or if all the correlations implied by Σ equal 1, then (4.4) holds. However, we will show that (4.4) is not true in general. The argument is similar to the one used in Section 3 and relies on the property (3.3) of the Cauchy distribution. Again we let $n = 2$ and parametrize the inverse of Σ as in (3.4).

Let $V = w_1 X_1\big(Y_1^{-2}\big) + w_2 X_2\big(Y_2^{-2}\big)$, with $w_1 \in [0, 1]$ and $w_2 = 1 - w_1$. Let g_V be the density of V. If f_{Y_1,Y_2} is the joint density of (Y_1, Y_2), then by (4.1),

$$g_V(v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_{y_1,y_2}(v) f_{Y_1,Y_2}(y_1, y_2)\, dy_1 dy_2, \tag{4.5}$$

where h_{y_1,y_2} is the density of a centered normal random variable with variance

$$\frac{w_1^2}{y_1^2(1-\theta^2)} - \frac{2w_1w_2\theta}{\max(y_1^2, y_2^2)(1-\theta^2)} + \frac{w_2^2}{y_2^2(1-\theta^2)}.$$

The part of the integral in (4.5) over the range $|y_1| < |y_2|$ can be written in the form

$$\begin{aligned}
&\int_{-1}^{1}\int_{-\infty}^{\infty} \frac{(1-\theta^2)|x| y_2^2}{(2\pi)^{3/2}(w_1^2 - 2w_1w_2\theta x^2 + w_2^2x^2)^{1/2}}\cdot \\
&\qquad \exp\left\{-\frac{(1-\theta^2)x^2y_2^2 l^2}{2(w_1^2 - 2w_1w_2\theta x^2 + w_2^2x^2)} - \frac{1}{2}(x^2y_2^2 + 2\theta x|y_2|y_2 + y_2^2)\right\} dy_2 dx \\
&= \int_0^1 \frac{(1-\theta^2)(w_1^2 - 2w_1w_2\theta x^2 + w_2^2x^2)x}{2\pi\left((w_1^2 - 2w_1w_2\theta x^2 + w_2^2x^2)(x^2 + 2\theta x + 1) + (1-\theta^2)x^2v^2\right)^{3/2}} dx \\
&\quad + \int_0^1 \frac{(1-\theta^2)(w_1^2 - 2w_1w_2\theta x^2 + w_2^2x^2)x}{2\pi\left((w_1^2 - 2w_1w_2\theta x^2 + w_2^2x^2)(x^2 - 2\theta x + 1) + (1-\theta^2)x^2v^2\right)^{3/2}} dx.
\end{aligned}$$

Computing in an identical manner the part of the integral in (4.5) over the range $|y_2| < |y_1|$ gives

$$\begin{aligned}
g_V(v) &= \int_0^1 \frac{(1-\theta^2)(w_1^2 - 2w_1w_2\theta x^2 + w_2^2x^2)x}{2\pi\left((w_1^2 - 2w_1w_2\theta x^2 + w_2^2x^2)(x^2 + 2\theta x + 1) + (1-\theta^2)x^2v^2\right)^{3/2}} dx \\
&\quad + \int_0^1 \frac{(1-\theta^2)(w_1^2 - 2w_1w_2\theta x^2 + w_2^2x^2)x}{2\pi\left((w_1^2 - 2w_1w_2\theta x^2 + w_2^2x^2)(x^2 - 2\theta x + 1) + (1-\theta^2)x^2v^2\right)^{3/2}} dx \\
&\quad + \int_0^1 \frac{(1-\theta^2)(w_2^2 - 2w_1w_1\theta x^2 + w_1^2x^2)x}{2\pi\left((w_2^2 - 2w_1w_2\theta x^2 + w_1^2x^2)(x^2 + 2\theta x + 1) + (1-\theta^2)x^2v^2\right)^{3/2}} dx \\
&\quad + \int_0^1 \frac{(1-\theta^2)(w_2^2 - 2w_1w_2\theta x^2 + w_1^2x^2)x}{2\pi\left((w_2^2 - 2w_1w_2\theta x^2 + w_1^2x^2)(x^2 - 2\theta x + 1) + (1-\theta^2)x^2v^2\right)^{3/2}} dx.
\end{aligned} \tag{4.6}$$

We will prove that (3.6) still holds, independently of θ in (3.4) and independently of the weights (w_1, w_2). Assuming that this is the case, the claim that g_V cannot be a Cauchy density for all choices of θ and (w_1, w_2) will, as in Section 3, follow once we show that, for any $0 < w_1 < 1$, the function $g_V(0)$ has a strictly positive derivative in θ at $\theta = 0$. By (4.6),

$$\begin{aligned}
g_V(0) &= \int_0^1 \frac{(1-\theta^2)(w_1^2 - 2w_1w_2\theta x^2 + w_2^2x^2)x}{2\pi(w_1^2 - 2w_1w_2\theta x^2 + w_2^2x^2)^{3/2}(x^2 + 2\theta x + 1)^{3/2}} dx \\
&\quad + \int_0^1 \frac{(1-\theta^2)(w_1^2 - 2w_1w_2\theta x^2 + w_2^2x^2)x}{2\pi(w_1^2 - 2w_1w_2\theta x^2 + w_2^2x^2)^{3/2}(x^2 - 2\theta x + 1)^{3/2}} dx \\
&\quad + \int_0^1 \frac{(1-\theta^2)(w_2^2 - 2w_1w_1\theta x^2 + w_1^2x^2)x}{2\pi(w_2^2 - 2w_1w_2\theta x^2 + w_1^2x^2)^{3/2}(x^2 + 2\theta x + 1)^{3/2}} dx \\
&\quad + \int_0^1 \frac{(1-\theta^2)(w_2^2 - 2w_1w_2\theta x^2 + w_1^2x^2)x}{2\pi(w_2^2 - 2w_1w_2\theta x^2 + w_1^2x^2)^{3/2}(x^2 - 2\theta x + 1)^{3/2}} dx.
\end{aligned}$$

It is elementary that one may differentiate with respect to θ inside the integrals to obtain

$$\left.\frac{\partial g_V(0)}{\partial \theta}\right|_{\theta=0} = \int_0^1 \frac{w_1 w_2 x^3}{\pi(w_1^2 + w_2^2 x^2)^{3/2}(x^2+1)^{3/2}}\,dx + \int_0^1 \frac{w_1 w_2 x^3}{\pi(w_2^2 + w_1^2 x^2)^{3/2}(x^2+1)^{3/2}}\,dx > 0,$$

as promised.

It remains to prove (3.6). We use (4.6) and we write in the obvious notation

$$v^2 g_V(v) = I_1(v) + I_2(v) + I_3(v) + I_4(v).$$

We have, as $v \to \infty$,

$$\begin{aligned} I_1(v) \sim& (1-\theta^2)w_1^2 \int_0^1 \frac{x v^2}{2\pi(w_1^2 + (1-\theta^2)v^2x^2)^{3/2}}\,dx \\ =& w_1 \int_0^{v(1-\theta^2)^{1/2}/w_1} \frac{x}{2\pi(x^2+1)^{3/2}}\,dx \to w_1 \int_0^\infty \frac{x}{2\pi(x^2+1)^{3/2}}\,dx = w_1/(2\pi). \end{aligned}$$

Similarly, as $v \to \infty$,

$$I_2(v) \to w_1/(2\pi), \quad I_3(v) \to w_2/(2\pi), \quad I_4(v) \to w_2/(2\pi),$$

and (3.6) follows from the fact that $w_1 + w_2 = 1$.

In summary, (4.4) does not hold for all values of the parameters. We prove the stronger contrary result that, for any choice of (w_1, w_2) with $0 < w_1 < 1$, $w_1 + w_2 = 1$, the sum on the left side of (4.4) does not have a Cauchy distribution for all θ in an arbitrarily small neighborhood of the origin.

5. Conclusions

Two natural transformations of multivariate normal vectors considered here and the similar transformation Pillai and Meng (2016) considered all result in a Cauchy distribution if the covariance matrix of the normal ingredients is diagonal, or if all the correlations implied by the covariance matrix equal 1. However, the transformation (2.2) considered by Pillai and Meng (2016) produces a Cauchy distribution regardless of the normal covariance matrix. By contrast, the transformations (3.2) and (4.4) do not always produce a Cauchy distribution. This shows that the mysteries of the connections between the normal laws and the Cauchy laws remain to be understood. The result of Pillai and Meng (2016) exhibits a family of laws in $\mathbb{R}^n$ with standard Cauchy marginals that share with a subclass of multivariate Cauchy laws (1.3) the property that any convex linear combination of the coordinates is again a standard Cauchy. Since the two transformations we have considered lack this property, the correlations between jointly normal random variables are not always overwhelmed by the heaviness of the marginal tails. Future work must clarify the extent of the phenomenon described by Drton and Xiao (2016) and Pillai and Meng (2016).

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