SIAM J. MATRIX ANAL. APPL.
Vol. 12, No. 3, pp. 592–596, July 1991

017

PERTURBATION THEORY OF A NONLINEAR GAME OF VON NEUMANN*

EZIO MARCHI†, JORGE A. OVIEDO†, AND JOEL E. COHEN‡

Abstract. Von Neumann and others considered a two-person zero-sum game with nonlinear payoff function x^TAy/x^TBy, where A and B are $m \times n$ matrices, x^T is the row m-vector strategy of the maximizing player (player 1), and y is the column n-vector strategy of the minimizing player (player 2). This game is defined to be completely mixed if every solution (or optimal strategy) (x, y) is such that all elements of x and all elements of y are positive. In such a game, it is supposed that the matrices A and B are infinitesimally perturbed by matrices of perturbations, i.e., multiple elements of each matrix are perturbed simultaneously. The effect of such perturbations on the solution and value of the game is calculated.

Key words. zero-sum game, two-person game, nonlinear game, perturbation theory, von Neumann model, economic model, stochastic game

AMS(MOS) subject classifications. primary 90D05; secondary 90A16

1. Introduction. This paper develops the perturbation theory of a finite, two-person, zero-sum game with a nonlinear payoff function proposed by von Neumann [13] in a model of economic growth. Subsequent development of the model has been synthesized by Morgenstern and Thompson [11]. The same payoff function appears in a special case of a stochastic game proposed by Shapley [12]. Because the game has more than economic interpretations, we shall not emphasize the economic view of the game nor restrict our assumptions to those that might be plausible in an economic application.

Von Neumann's game has m pure strategies for player 1, the maximizing player, and n pure strategies for player 2, the minimizing player, where $1 \leqq m, n < \infty$. The strategy of player 1 is specified by a row m-vector x^T, where x_i is the probability that player 1 chooses pure strategy i, for $i = 1, \cdots, m$. The strategy of player 2 is specified by a column n-vector y, where y_j is the probability that player 2 chooses pure strategy j, for $j = 1, \cdots, n$. The payoff function of the game, that is, the amount of money player 2 must pay player 1 if player 1 has strategy x^T and player 2 has strategy y, is x^TAy/x^TBy, where A and B are real $m \times n$ matrices. This payoff function is defined (though possibly equal to $+\infty$) provided its numerator and denominator are not simultaneously equal to zero; additional conditions will be provided to assure that the payoff function is always defined. As there does not appear to be a standard name for this game, we shall call it a *rational* game specified by (A, B), because the payoff function is a ratio of linear forms.

Marchi [8] extended and generalized the equilibrium points of a rational game to an n-person game with a rational payoff function. Marchi [7] and Marchi, Tarazaga, and Elorza [9] applied such results to expanding economies.

The perturbation theory of a game describes how small variations in the parameters of the payoff function affect the solution and value of the game. The perturbation theory

* Received by the editors March 29, 1989; accepted for publication (in revised form) December 15, 1989. This work was supported in part by National Science Foundation grants BSR 84-07461 and BSR 87-05047, and by the hospitality of Mr. and Mrs. William T. Golden.

† Instituto de Matemática Aplicada, Universidad Nacional de San Luis, 5700 San Luis, República Argentina (banyc!atina!imasl!oviedo@uunet.uu.net).

‡ Rockefeller University, 1230 York Avenue, Box 20, New York, New York 10021 (cohen@rockvax.rockefeller.edu). This work began during this author's visit to Argentina in 1987, arranged through the Sistema Para el Apoyo a la Investigación y Desarrollo de la Ecología en la República Argentina, and supported by CONICET, Argentina.

of games in general, and of rational games in particular, is of practical interest for both estimation and control. The value of a rational game has an economic interpretation as the asymptotic rate of change (growth or decrease) of an economy. In economic applications of the game, the matrices A (the output matrix) and B (the input matrix) must be estimated from data. The first derivative of the value with respect to the elements of A and B indicates the value's sensitivity to errors in the values of these elements, and therefore indicates which elements should be measured with greatest precision. Kuhn and Tucker [6, p. viii] recognized the importance of perturbation theory for control in their introduction to the work of Mills [10]: "This study promises practical application whenever these parameters [the matrix elements] can be controlled or altered since it indicates which changes will have a beneficial effect on the value."

To our knowledge, the perturbation theory of rational games has not been studied before, except in the linear special case when $B = J_{m,n}$, where $J_{m,n}$ is the $m \times n$ matrix with every element equal to one. In this case, a rational game reduces to an ordinary two-person zero-sum matrix game. Previous studies of the perturbation theory of zero-sum matrix games are reviewed by Cohen [1] and Cohen, Marchi, and Oviedo [3].

We now establish notation and state some results which are mostly standard or readily proved.

Let $P_n = \{x \in R^n: x_i \geqq 0,\ i = 1, 2, \cdots, n, \text{ and } \sum_{i=1}^n x_i = 1\}$ and $P_n^+ = \{x \in P_n: x_i > 0,\ i = 1, \cdots, n\}$. Vectors are assumed to be column vectors, and the superscript T denotes transpose.

Given two real $m \times n$ matrices A and B we say that a pair of vectors $(\bar{x}^T, \bar{y}) \in P_m \times P_n$ is a *solution of the matricial problem* if

$$(\bar{x}^T A\bar{y})(x^T B\bar{y}) - (x^T A\bar{y})(\bar{x}^T B\bar{y}) \geqq 0 \quad \forall x \in P_m,$$

$$(\bar{x}^T Ay)(\bar{x}^T B\bar{y}) - (\bar{x}^T A\bar{y})(\bar{x}^T By) \geqq 0 \quad \forall y \in P_n.$$

To prove the existence of a solution to the matricial problem, we recall a result of Marchi [8] which is a special case of a theorem of Karamardian [5].

LEMMA. *Consider a real continuous function $\phi: \Sigma \times \Sigma \to \Re$ defined on the Cartesian square of Σ, a nonempty, compact, convex set in a Euclidean space. If $\phi(\cdot, \tau)$ is concave for any $\tau \in \Sigma$, then there exists a point $\bar{\sigma} \in \Sigma$ such that*

$$\phi(\bar{\sigma}, \bar{\sigma}) = \max_{\sigma \in \Sigma} \phi(\sigma, \bar{\sigma}).$$

THEOREM 1. *For any real $m \times n$ matrices A and B, a matricial problem has a solution.*

The theorem is easily proved by using the lemma. Essentially this theorem was known to von Neumann [13]. Under the conditions given by von Neumann [13], namely,

$$a_{ij} \geqq 0, \qquad b_{ij} \geqq 0, \qquad a_{ij} + b_{ij} > 0, \qquad i, j = 1, \cdots, n,$$

the solution of the matricial problem determines a solution or saddle point of the rational zero-sum game with payoff function $x^T Ay/x^T By$. In what follows, instead of von Neumann's assumptions we assume $B > 0$, i.e., every element b_{ij} of B is positive real. Shapley [12] considers the same assumption. Then a solution of the matricial problem satisfies

$$\frac{x^T A\bar{y}}{x^T B\bar{y}} \leqq \frac{\bar{x}^T A\bar{y}}{\bar{x}^T B\bar{y}} \leqq \frac{\bar{x}^T Ay}{\bar{x}^T By} \quad \forall x \in P_m, \quad \forall y \in P_n,$$

which is a saddle point of the rational game. Any saddle point determines the value v of the game $v = (\bar{x}^T A \bar{y})/(\bar{x}^T B \bar{y})$, and furthermore

$$v = \max_{x \in P_m} \min_{y \in P_n} (x^T Ay)/(x^T By) = \min_{y \in P_n} \max_{x \in P_m} (x^T Ay)/(x^T By).$$

Solutions are interchangeable. That is, if $(\bar{x}, \bar{y})$ and $(\tilde{x}, \tilde{y})$ are two saddle points of a rational game, then $(\bar{x}, \tilde{y})$ and $(\tilde{x}, \bar{y})$ are also saddle points. The proof is identical to the proof for ordinary zero-sum matrix games. Optimal strategies for both players may be defined as in matrix games. The set of optimal strategies of each player is a nonempty convex polyhedron.

A rational game is defined to be *completely mixed* (abbreviated "cm") when, for every solution (x, y), $x > 0$ and $y > 0$. When $B = J_{m,n}$, this definition reduces to Kaplansky's [4] definition of a completely mixed matrix game. Completely mixed rational games exist. For example, let $m = n$, $B = J_{n,n}$ and let A be a diagonal matrix with positive elements on the main diagonal. This rational game is an ordinary zero-sum matrix game, and Kaplansky [4] proved that it is cm.

In a rational game (A, B) with $B > 0$ (as we assume throughout), if (x, y) is a solution, then $(Ay)_i/(By)_i < v$ implies $x_i = 0$ and $(x^T A)_j/(x^T B)_j > v$ implies $y_j = 0$. Therefore, in a cm rational game specified by (A, B) with $B > 0$, if (x, y) is a solution, then $(Ay)_i/(By)_i = v$, for all $i = 1, \cdots, m$, and $(x^T A)_j/(x^T B)_j = v$, for all $j = 1, \cdots, n$. Equivalently, $Ay = vBy$ and $x^T A = vx^T B$. If, in addition, $A \neq 0$ and $A \geqq 0$ (i.e., each element a_{ij} of A is nonnegative real), then $v > 0$. The proofs of these remarks are straightforward and are omitted.

Let $\rho(A)$ denote the spectral radius (maximum modulus of the eigenvalues) of an $n \times n$ matrix A. Under certain conditions, there is a direct connection between the value of a cm rational game specified by (A, B) and the spectral radius of $A^{-1}B$.

PROPOSITION 1. *In a* cm *rational game specified by* (A, B) *with* $m = n$ *and* $B > 0$, *if* A *is nonsingular and* $A^{-1}B > 0$, *then* $v = 1/\rho(A^{-1}B) > 0$ *and, for every solution* (x, y), y *is unique and* $x^T B$ *is unique. These are the right and left positive eigenvectors of* $A^{-1}B$ *corresponding to the eigenvalue* $1/v$.

Proof. By Perron's theorem for positive matrices applied to $A^{-1}B$, $A^{-1}B$ has a unique positive right eigenvector in P_n^+. But from previous remarks, $y = vA^{-1}By$. As $y > 0$, $A^{-1}B > 0$, evidently $v > 0$ and $v^{-1}y = A^{-1}By$, so y must be that unique right eigenvector corresponding to the positive eigenvalue v^{-1} and there can exist no other $\eta \in P_n^+$ such that $A\eta = vB\eta$. Similarly, $x^T A(A^{-1}B) = vx^T B(A^{-1}B)$ or $(x^T B)v^{-1} = (x^T B)(A^{-1}B)$. □

This proposition has slightly stronger assumptions and arrives at slightly stronger conclusions than Theorem 5(a) of Cohen and Friedland [2].

While $x^T B$ is unique, under the assumptions of Proposition 1, it is clear that x^T need not be unique.

2. Perturbation theory. Let A, B, G, H be fixed $n \times n$ real matrices, $B > 0$, and for each real number α, define

$$L = L(\alpha) = A + \alpha G, \qquad M = M(\alpha) = B + \alpha H.$$

It is clear that if $B > 0$ and A is nonsingular and $A^{-1}B > 0$, then there exists a real number $r > 0$ such that, for all real α with $|\alpha| < r$, (i) $M(\alpha) > 0$, (ii) $L(\alpha)$ is nonsingular, (iii) $[L(\alpha)]^{-1}M(\alpha) > 0$, and (iv) $\rho([L(\alpha)]^{-1}M(\alpha))$ is analytic in α.

Define a rational game specified by (A, B) to be nonsingular if A and B are both $n \times n$ and both nonsingular.

PROPOSITION 2. *Suppose a nonsingular rational game specified by* (A, B) *is* cm, $B > 0$, *and* $A^{-1}B > 0$. *Then there exists a real number r such that if* $|\alpha| < r$, *the rational* *game specified by* $(L(\alpha), M(\alpha))$ *is nonsingular and* cm, *the value* $v(\alpha)$ *of that game is given by* $v(\alpha) = 1/\rho([L(\alpha)]^{-1}M(\alpha))$, *and the solution* $(x(\alpha), y(\alpha))$ *of that game is unique.*

In other words, for sufficiently small perturbations (measured by α), under the common assumptions of Propositions 1 and 2, the conclusions of Proposition 1 about the unperturbed rational game specified by (A, B) carry over to the perturbed rational game specified by $(L(\alpha), M(\alpha))$.

Proof. If A and B are nonsingular, then so are sufficiently small perturbations of A and B. Thus, the rational game specified by $(L(\alpha), M(\alpha))$ is nonsingular for small enough values of α. By Proposition 1, the game specified by (A, B) has solutions (x, y) such that y and $z^T = x^TB$ are unique. Because B^{-1} exists, $z^TB^{-1} = x^T$ is also unique. As (x, y) is the solution of a cm rational game, $x > 0$ and $x^T(AB^{-1}) = vx^T$, i.e., x^T is the left eigenvector of AB^{-1} corresponding to the eigenvalue v. Sufficiently small perturbations of $A = L(0)$ and $B = M(0)$ to $L(\alpha)$ and $M(\alpha)$, respectively, will result in a sufficiently small perturbation of v to $v(\alpha)$ such that the corresponding left eigenvector $x^T(\alpha)$ of $L(\alpha)[M(\alpha)]^{-1}$ remains positive and the corresponding right eigenvector $y(\alpha)$ of $[L(\alpha)]^{-1}M(\alpha)$ remains positive. That $(x(\alpha), y(\alpha))$ is unique is guaranteed because $y(\alpha)$ and $z^T(\alpha) = x^T(\alpha)M(\alpha)$ are unique by the Perron theorem and therefore $x^T(\alpha)$ is unique by the nonsingularity of $M(\alpha)$. Thus for small enough α, every solution of $(L(\alpha), M(\alpha))$ is positive, i.e., $(L(\alpha), M(\alpha))$ is cm. Proposition 1 then guarantees that $v(\alpha) = 1/\rho([L(\alpha)]^{-1}M(\alpha))$. $\square$

THEOREM 2. *In a nonsingular* cm *rational game specified by* (A, B) *with* $B > 0$ *and* $A^{-1}B > 0$, *let* A *be perturbed to* $A + \alpha G$ *and* B *be perturbed to* $B + \alpha H$. *Then there exists* $r > 0$ *such that, for* $|\alpha| < r$, $dv(\alpha)/d\alpha$ *exists. Moreover, evaluated at* $\alpha = 0$, *the derivative is*

$$\frac{dv(0)}{d\alpha} = \frac{x^T(G - vH)y}{x^TBy},$$

where (x, y) *and* v *are the solution and value of the original game specified by* (A, B).

Proof. The existence of the derivative follows from Proposition 2 and preceding remarks. Now use the chain rule. If $(x(\alpha), y(\alpha))$ and $v(\alpha)$ are the solution and value of the nonsingular cm rational game specified by $(L(\alpha), M(\alpha))$, then

$$\begin{aligned}
\frac{dv(\alpha)}{d\alpha} &= \frac{d}{d\alpha}\left(\frac{x^T(\alpha)L(\alpha)y(\alpha)}{x^T(\alpha)M(\alpha)y(\alpha)}\right) \\
&= \Bigg\{x^T(\alpha)M(\alpha)y(\alpha)\left[\frac{dx^T(\alpha)}{d\alpha}L(\alpha)y(\alpha) + x^T(\alpha)Gy(\alpha) + x^T(\alpha)L(\alpha)\frac{dy(\alpha)}{d\alpha}\right] \\
&\qquad - (x^T(\alpha)L(\alpha)y(\alpha))\Bigg[\frac{dx^T(\alpha)}{d\alpha}M(\alpha)y(\alpha) + x^T(\alpha)Hy(\alpha) \\
&\qquad\qquad + x^T(\alpha)M(\alpha)\frac{dy(\alpha)}{d\alpha}\Bigg]\Bigg\} \Big/ (x^T(\alpha)M(\alpha)y(\alpha))^2 \\
&= \Bigg[\frac{dx^T(\alpha)}{d\alpha}(L(\alpha) - v(\alpha)M(\alpha))y(\alpha) \\
&\qquad + x^T(\alpha)(G - v(\alpha)H)y(\alpha) \\
&\qquad + x^T(\alpha)(L(\alpha) - v(\alpha)M(\alpha))\frac{dy(\alpha)}{d\alpha}\Bigg] \Big/ (x^T(\alpha)M(\alpha)y(\alpha)).
\end{aligned}$$

Because the game specified by $(L(\alpha), M(\alpha))$ is cm and $M(\alpha) > 0$, it follows that $(L(\alpha) - v(\alpha)M(\alpha))y(\alpha) = 0$ and $x^T(\alpha)(L(\alpha) - v(\alpha)M(\alpha)) = 0$. Then taking $\alpha = 0$ gives the claimed formula. □

Under the assumptions of Theorem 2, the derivatives $d^2v(0)/d\alpha^2$, $dx^T(0)/d\alpha$ and $dy(0)/d\alpha$ exist and satisfy

$$\begin{aligned}\frac{d^2v(0)}{d\alpha^2} = \frac{dx^T(0)}{d\alpha}\left(G - vH - \frac{dv(0)}{d\alpha}B\right)\frac{y}{x^TBy}\\ + \frac{x^T}{x^TBy}\left(G - vH - \frac{dv(0)}{d\alpha}B\right)\frac{dy(0)}{d\alpha} - 2\frac{x^THy}{x^TBy}\frac{dv(0)}{d\alpha},\end{aligned}$$

$$\frac{dx^T(0)}{d\alpha}(A - vB) = \left(\frac{dv(0)}{d\alpha}\right)x^TB - x^T(G - vH),$$

$$(A - vB)\frac{dy(0)}{d\alpha} = \left(\frac{dv(0)}{d\alpha}\right)By - (G - vH)y.$$

These formulas follow from applying the chain rule to, respectively, the formula for $dv(0)/d\alpha$ and the identities

$$x^T(\alpha)[L(\alpha) - v(\alpha)M(\alpha)] = 0, \qquad [L(\alpha) - v(\alpha)M(\alpha)]y^T(\alpha) = 0.$$

It is not difficult to verify that when $B = J_{n,n}$ and $H = 0$, the preceding formulas reduce to those found for ordinary zero-sum matrix games by Mills [10] and Cohen [1].

A task for the future is to derive perturbation results similar to the preceding under weaker or different conditions from those assumed in Theorem 2.

Acknowledgments. Several careful referees improved this paper.

REFERENCES

[1] J. E. COHEN, *Perturbation theory of completely mixed matrix games*, Linear Algebra Appl., 79 (1986), pp. 153–162.

[2] J. E. COHEN AND S. FRIEDLAND, *The game-theoretic value and the spectral radius of a nonnegative matrix*, Proc. Amer. Math. Soc., 93 (1985), pp. 205–211.

[3] J. E. COHEN, E. MARCHI, AND J. A. OVIEDO, *Perturbation theory of completely mixed bimatrix games*, Linear Algebra Appl., 114/115 (1989), pp. 169–180.

[4] I. KAPLANSKY, *A contribution to von Neumann's theory of games*, Ann. of Math., 46 (1945), pp. 474–479.

[5] S. KARAMARDIAN, *Generalized complementarity problem*. J. Optim. Theory Appl., 8 (1971), pp. 161–168.

[6] H. W. KUHN AND A. W. TUCKER, EDS., *Preface*, Linear Inequalities and Related Systems, Princeton University Press, Princeton, NJ, 1956.

[7] E. MARCHI, *El modelo de crecimiento de von Neumann para un número arbitrario de países*, Rev. Un. Mat. Argentina, 29 (1979), pp. 85–95.

[8] ———, *Equilibrium points of rational N-person games*, J. Math. Anal. Appl., 54 (1976), pp. 1–4.

[9] E. MARCHI, P. TARAZAGA, AND E. ELORZA, *Further topics in von Neumann growth model*, Portugal. Math., 42 (1983/4), pp. 255–264.

[10] H. D. MILLS, *Marginal values of matrix games and linear programs*, in Linear Inequalities and Related Systems, H. W. Kuhn and A. W. Tucker, eds., Princeton University Press, Princeton, NJ, 1956, pp. 183–193.

[11] O. MORGENSTERN AND G. L. THOMPSON, *Mathematical Theory of Expanding and Contracting Economies*, D. C. Heath, Lexington, MA, 1976.

[12] L. S. SHAPLEY, *Stochastic games*, Proc. Nat. Acad. Sci. U.S.A., 39 (1953), pp. 1095–1100.

[13] J. VON NEUMANN, *Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes*, Ergebnisse eines Mathematischen Kolloquiums, 8 (1935/6), pp. 73–83. Translated by G. Morgenstern in Review of Economic Studies, 13 (1945/6), pp. 1–9. Reprint of English translation in Collected Works 6, pp. 29–37, Macmillan, New York, 1963.