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CONVEXITY PROPERTIES OF GENERALIZATIONS OF THE ARITHMETIC-GEOMETRIC MEAN

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ABSTRACT

In the eighteenth century, Landen, Lagrange and Gauss studied a function of two positive real numbers that has become known as the arithmetic-geometric mean (AGM). In the nineteenth century, Borchardt generalized the AGM to a function of any 2^n $(n = 1, 2, 3, \ldots)$ positive real numbers. In this paper, we generalize the AGM to a function of any even number of positive real numbers. If M(a, b) is the original AGM then M(a, b) is concave in the pair (a, b) of positive numbers and log $M(e^\alpha, e^\beta)$ is convex in the pair (α, β) of real numbers; all our generalizations of the AGM behave similarly. We generalize this analysis extensively.

If a and b are positive real numbers, define a new pair (a_1, b_1) of positive reals by

$$(a_1, b_1) = f(a, b) = \left(\frac{a+b}{2}, [ab]^{1/2}\right). \tag{1}$$

If f^k denotes the kth iterate of f, there is a positive number $\lambda = \lambda(a, b)$ such that

$$\lim_{k \to \infty} f^k(a, b) = (\lambda, \lambda). \tag{2}$$

The number λ is usually called the arithmetic-geometric mean or AGM of a and b and is sometimes written M(a, b). Landen, Lagrange and Gauss (see [2,7] for references) independently proved that

$$\lambda(a, b) = \frac{\pi}{2I(a, b)} \tag{3}$$

where

$$I(a, b) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{[a^2 \cos^2\theta + b^2 \sin^2\theta]^{1/2}}. \tag{4}$$

A nice proof of (3) is given by Carlson [5].

Recall that a map g from a convex subset D of a vector space X to the reals is called "convex" if $g((1-t)x + ty) \leq (1-t)g(x) + tg(y)$ for all $x, y \in D, 0 \leq t \leq 1$. The map g is "concave" if -g is convex. Define $K = \{x \in \Re^n : x_j \geq 0 \text{ for } 1 \leq j \leq n\}$ and $\overset{o}{K}$ to be the interior of K, i.e., the set of real n vectors with positive elements. More generally, if K is a cone in a Banach space and if $f : \overset{o}{K} \to \overset{o}{K}$ is a suitable map, it may happen that for every $x \in \overset{o}{K}$ there exists a fixed point $g(x) \in \overset{o}{K}$ of f such that $\lim f^k(x) = g(x)$: see Theorem 1 below or Section 3 of [12] for some general results of this type. For $x \in \Re^n$ and $y \in \overset{o}{K}$, define $e^x = (e^{x_1}, e^{x_2}, \ldots, e^{x_n})$ and $\log y = (\log y_1, \log y_2, \ldots, \log y_n)$. We shall give general conditions under which $y \to g(y)$ is concave or convex or $x \to \log g(e^x)$ is convex. The purpose of this note is to prove very general convexity results of this type, by elementary arguments, for large classes of examples of interest. A very special corollary of these results for the AGM is that the map $(a, b) \to \lambda(a, b)$ is concave and $(\alpha, \beta) \to \log \lambda(e^\alpha, e^\beta)$ is convex, and even the latter result may be new.

If a, b, c, d are positive reals, Borchardt (see [2,4]) defined a map f by

$$f(a, b, c, d) = \left(\frac{a+b+c+d}{4}, \frac{[ab]^{1/2} + [cd]^{1/2}}{2}, \frac{[ac]^{1/2} + [bd]^{1/2}}{2}, \frac{[ad]^{1/2} + [bc]^{1/2}}{2} \right) \tag{5}$$

and proved (this is the easy part of his work) that

$$\lim_{k \to \infty} f^k(a, b, c, d) = (\lambda, \lambda, \lambda, \lambda), \quad \lambda > 0. \tag{6}$$

Borchardt defined an analogous map whenever the number of variables is a power of 2.

We now offer a new natural generalization of the map (5) whenever n, the number of variables, is an even integer, n = 2m. To do this we recall a special case of a much deeper result of Baranyai [3] concerning partitions of a finite set into subsets with j elements. Let X = {1, 2, . . . , n} and let E denote the collection of subsets with exactly two elements; so E has n(n-1)/2 elements. If n = 2m, then one can partition E into n-1 disjoint sets F_j, $1 \leq j \leq n-1$, each containing m elements, such that if, for some j, A, B $\in F_j$, then $A \cap B = \emptyset$ if $A \neq B$; and such that

$$X = \bigcup_{A \in F_j} A, \quad \text{for each } j .$$

If n = 2m and E is partitioned into n-1 subsets F_j as above, define

$$\phi_{F_j}(x) = \frac{1}{n} \sum_{\substack{1 \leq i,k \leq n \\ \{i,k\} \in F_j}} [x_i x_k]^{1/2}, \quad \text{for } x \in K . \tag{7}$$

Define a map $f : \overset{o}{K} \rightarrow \overset{o}{K}$ (dependent on the above partition of E) by

$$f_1(x) = \frac{1}{n} \sum_{k=1}^{n} x_i , \tag{8}$$

$$f_j(x) = \phi_{F_{j-1}}(x) \quad \text{for } 2 \leq j \leq n , \tag{9}$$

where $f_j(x)$ denotes the jth component of f(x). We shall refer to such a map as a "Borchardt map." For example, if n=6, a Borchardt map is given by

$$f_1(x) = \frac{1}{6}(x_1 + x_2 + x_3 + x_4 + x_5 + x_6),$$

$$f_2(x) = \frac{1}{3}([x_1x_2]^{1/2} + [x_3x_4]^{1/2} + [x_5x_6]^{1/2}),$$

$$f_3(x) = \frac{1}{3}([x_1x_3]^{1/2} + [x_2x_5]^{1/2} + [x_4x_6]^{1/2}),$$

$$f_4(x) = \frac{1}{3}([x_1x_4]^{1/2} + [x_2x_6]^{1/2} + [x_3x_5]^{1/2}),$$

$$f_5(x) = \frac{1}{3}([x_1x_5]^{1/2} + [x_3x_6]^{1/2} + [x_2x_4]^{1/2}),$$

$$f_6(x) = \frac{1}{3}([x_1x_6]^{1/2} + [x_4x_5]^{1/2} + [x_2x_3]^{1/2}).$$

According to Corollary 2 below, if $f:\overset{o}{K} \to \overset{o}{K}$ is a Borchardt map, $\lim_{k \to \infty} f^k(x) = \lambda(x)(1, 1, \ldots, 1)$ for every $x \in \overset{o}{K}$, where $\lambda(x) > 0$ and $x \to \lambda(x)$ is real analytic. Again one can ask about concavity or convexity properties of the map $x \to \lambda(x)$.

There are many examples of "means and their iterates": see [1, 2, 5-8, 10-15] and the references there. In order to handle these examples in a reasonably unified way, we shall need a general framework. If X is a Hausdorff, topological vector space over the real numbers, a subset C of X will be called a cone (with vertex at 0) if C is closed and convex, $tC \subset C$ for all $t > 0$, and $x \in C - \{0\}$ implies that $-x \notin C$. An example is provided by $K = \{x \in \mathfrak{R}^n : x_j \geq 0 \text{ for } 1 \leq i \leq n\}$, which we shall call the standard cone in $\mathfrak{R}^n$. A cone induces a partial ordering on x by $x \leq y$ if and only if $y - x \in C$. If x and y are elements of C, x and y will be called "comparable" if there exist strictly positive scalars α and β such that $\alpha x \leq y \leq \beta x$. Comparability is an equivalence relationship and divides C into disjoint equivalence classes called "components of C". If $u \in C - \{0\}$, we shall define C_u by

$$C_u = \{x \in C : x \quad \text{is comparable to} \quad u\}\ . \tag{10}$$

Note that if C has nonempty interior and $u \in \overset{o}{C}$, then $C_u = \overset{o}{C}$. In the standard cone in $\mathfrak{R}^n$, C_u is all the vectors in C with positive elements at the same positions as those of u.

Suppose that X_j, $j = 1, 2$, is a Hausdorff topological real vector space with cone C_j. If D is a subset of X_1, a map $f : D \to X_2$ is called "order-preserving" if $f(x) \leq_2 f(y)$ for all $x, y \in D$ such that $x \leq_1 y$. Here $\leq_1$ denotes the partial ordering induced by C_1 and $\leq_2$ that induced by C_2. If $X_1 = X_2$ we shall usually have $C_1 = C_2$. If D is a convex subset of X_1, a map $g : D \to X_2$ will be called "concave" if $g((1-t)x + ty) \geq (1-t)g(x) + tg(y)$ for all $x, y \in D$ and real numbers t with $0 \leq t \leq 1$; g will be called convex if $-g$ is concave.

Now suppose that C is a cone in a Hausdorff topological real vector space X, C_v is a component of C and $f : C_v \to C_v$ is a map. Assume that for every $x \in C_v$ (or for every x in some convex open subset G of C_v) there exists $u(x) \in C_v$ such that

$$\lim_{k \to \infty} f^k(x) = u(x)\ . \tag{11}$$

We are interested in concavity and convexity properties of x → u(x). In many examples, the vector u(x) in (11) is always a positive multiple of a fixed vector $u \in C_v$, $u(x) = \lambda(x)u$; and in this case one can ask about concavity and convexity properties of $x \to \lambda(x)$.

The existence of a limit as in (11) is a strong assumption, but there are many examples for which the existence of such a limit has been established: see [2, 6, 14], Section 3 of [12] and [15]. We mention explicitly a special case of Theorem 3.2 in [12].

Theorem 1. (See Theorem 3.2 in [12].) Let C be a cone with nonempty interior $\overset{o}{C}$ in a finite dimensional Banach space X. Assume that $f: \overset{o}{C} \to \overset{o}{C}$ is order-preserving (with respect to the partial order induced by C) and homogeneous of degree one (so $f(tx) = tf(x)$ for all $x \in \overset{o}{C}$ and t > 0). Assume that f(u) = u for some $u \in \overset{o}{C}$, that f is continuously Fréchet differentiable on an open neighborhood of u, and that there exists an integer $m \geq 1$ such that $L^m(C - \{0\}) \subset \overset{o}{C}$, where L = f′(u) is the Fréchet derivative of f at u. Then for every $x \in \overset{o}{C}$ there exists λ(x) > 0 such that

$$\lim_{k \to \infty} \| f^k(x) - \lambda(x)u \| = 0 .$$

The map $x \to \lambda(x)$ is continuous on $\overset{o}{C}$ and continuously differentiable on an open neighborhood of u. If $u^* = \lambda'(u)$, the Fréchet derivative of λ at u, then $u^*(u) = 1$ and $L^*(u^*) = u^*$. If f is C^k (real analytic) on $\overset{o}{C}$, then $x \to \lambda(x)$ is C^k (real analytic) on $\overset{o}{C}$.

Related theorems in which f is not necessarily order-preserving are given in Section 3 of [12] and [15].

If K is the standard cone in $\mathfrak{R}^n$, then L is the Jacobian matrix of f at u and L has all nonnegative entries. The assumptions of the theorem amount to the assumption that L is "primitive," i.e., L^m has all positive entries for some positive integer m.

It will be useful to recall the definition of a class M of maps of the standard cone K in $\mathfrak{R}^n$ into itself. The class M has been extensively studied in [12, 13] and includes many examples of generalized means. If σ is a probability vector in K (so $\sum_{i=1}^{n} \sigma_i = 1$) and r is a real number, define a map $M_{r\sigma}$: $\overset{o}{K} \to (0, \infty)$ by

$$M_{r\sigma}(x) = \left(\sum_{i=1}^{n} \sigma_i x_i^r \right)^{1/r} . \tag{12}$$

If r = 0, define

$$M_{0\sigma}(x) = \prod_{i=1}^{n} x_i^{\sigma_i} = \lim_{r\to 0} M_{r\sigma}(x)\,. \tag{13}$$

For each i, $1 \le i \le n$, let Γ_i be a finite collection of ordered pairs (r, σ), where $r \in \Re$ and σ is a probability vector in K; and for $1 \le i \le n$ and $(r, \sigma) \in \Gamma_i$ suppose that $c_{ir\sigma}$ is a given positive number. Define a map $f : \overset{o}{K} \to \overset{o}{K}$ by

$$f_i(x) = \text{the } i \text{ th component of} \quad f(x) = \sum_{(r,\sigma)\in\Gamma_i} c_{ir\sigma} M_{r\sigma}(x)\,. \tag{14}$$

If $f : \overset{o}{K} \to \overset{o}{K}$ can be written as in (14) we shall say that $f \in M$. If $f \in M$ and $f_i(x)$ can be expressed as in (14) in such a way that $r \ge 0$ for all $(r, \sigma) \in \Gamma_i$, $1 \le i \le n$, we shall write $f \in M_+$; if f can be written so that $r < 0$ for all $(r, \sigma) \in \Gamma_i$, $1 \le i \le n$, we shall write $f \in M_-$. Linear maps in M lie in $M_+ \cap M_-$ because, if δ_{jk} is the Kronecker delta,

$$\sum_{(r,\sigma)\in\Gamma_i} c_{ir\sigma} \sum_{j=1}^{n} \sigma_j x_j \quad = \quad \sum_{(r,\sigma)\in\Gamma_i} \sum_{j=1}^{n} \quad c_{ir\sigma}\sigma_j \left(\sum_{k=1}^{n} \delta_{jk} x_k^{-1} \right)^{-1}$$

and the left side is in M_+ while the right is in M_-. We define M (M_+, M_-) to be the smallest set of maps $f : \overset{o}{K} \to \overset{o}{K}$ which is closed under composition of functions and addition of functions and contains M (M_+, M_-). One can prove (see [14] and Section 2 of [13]) that if $f \in M$, then f is order-preserving, homogeneous of degree one, C^∞ (in fact, real analytic) on $\overset{o}{K}$, and extends continuously to K. In particular, Borchardt maps are elements of M_+ and are order-preserving and homogeneous of degree one. We shall apply our theorems to functions $f \in M$.

For completeness, we begin with some easy lemmas, the proofs of which are omitted.

Lemma 1. Let D_i be a convex subset of a Hausdorff, topological real vector space X_i, i = 1, 2, and suppose that X_3 is also a Hausdorff topological real vector space. Assume that C_i, i = 2, 3, is a cone in X_i and that $g : D_1 \to D_2$ is a concave (respectively, convex) map with respect to the ordering induced by C_2 and $f : D_2 \to C_3$ is concave (respectively, convex) and order-preserving with respect to the orderings induced by C_2 and C_3. Then $h = f \cdot g$ is concave (respectively, convex) and h is order-preserving if g is order-preserving.

Lemma 2. Suppose that X is a Hausdorff topological real vector space and that C is a cone in X. Assume that D is a convex subset of X and that $g_k : D \rightarrow X$, $1 \leq k < \infty$, is a concave (respectively, convex) map with respect to the partial ordering induced by C. Assume that for every $x \in D$ one has

$$\lim_{k \to \infty} g_k(x) = g(x)\,. \tag{15}$$

Then the map g is concave (respectively, convex).

If g_k in Lemma 2 is order-preserving for all $k \geq 1$, one easily can prove that g is order-preserving.

Theorem 2. Let C be a cone in a Hausdorff topological real vector space X and for $v \in C\text{-}\{0\}$, let C_v be as in (10). Assume that $f : C_v \rightarrow C_v$ is order-preserving and concave (respectively, convex) and that for every $x \in C_v$, there exists $u(x) \in C_v$ such that

$$\lim_{k \to \infty} f^k(x) = u(x)\,. \tag{16}$$

Then the map $x \rightarrow u(x)$ is concave (respectively, convex) and order-preserving.

Proof. Define $g_k(x) = f^k(x)$ and $g(x) = u(x)$. Repeated application of Lemma 1 implies that g_k is concave and order-preserving. The conclusion of the theorem then follows from Lemma 2. []

Theorem 2 is of interest only if one can find examples of functions f which are order-preserving, concave (or convex) and satisfy (16). The next theorem gives a start in this direction.

Theorem 3. Let the notation and the assumptions be as in Theorem 1. In addition, assume that $f : \overset{o}{C} \rightarrow \overset{o}{C}$ is concave (respectively, convex). Then the map $x \rightarrow \lambda(x)$ is concave (respectively, convex).

Proof. Theorems 1 and 2 imply that $x \rightarrow \lambda(x)u$ is concave (convex). By the Hahn-Banach theorem there exists a continuous linear functional ψ which is nonnegative on C and satisfies $\psi(u) = 1$. Because ψ is concave and order-preserving, the map $x \rightarrow \psi(\lambda(x)u) = \lambda(x)$ is concave. []

It remains to give some examples. The next lemma is a classical result.

Lemma 3. (See [9].) Let K denote the standard cone in $\Re^n$, r a real number and $\sigma \in K$ a probability vector. If $r \leq 1$, the map $x \in \overset{o}{K} \rightarrow M_{r\sigma}(x)$ is concave; and if $r \geq 1$, the map is convex.

Now define M_1 to be the collection of functions $f \in M$ such that for $1 \leq i \leq n$, $f_i(x)$ can be represented as in (14) so that $r \leq 1$ for all $(r, \sigma) \in \Gamma_i$, $1 \leq i \leq n$. Define M_1 to be the smallest set of functions $f : \overset{o}{K} \to \overset{o}{K}$ such that M_1 contains M_1 and M_1 is closed under addition and composition of functions.

Lemma 4. If $f \in M_1$, f is homogeneous of degree 1, order-preserving and concave.

Proof. We have already noted that if $f \in M \supset M_1$, f is homogeneous of degree one and order-preserving. It remains to prove that f is concave. Let A denote the set of maps $f : \overset{o}{K} \to \overset{o}{K}$ which are homogeneous of degree one, order-preserving and concave. Lemma 3 implies that $M_1 \subset A$, and Lemma 1 implies that A is closed under composition. The proof that A is closed under addition is also easy and left to the reader. The minimality of M_1 now implies that $M_1 \subset A$. []

Corollary 1. Let K denote the standard cone in $\mathfrak{R}^n$ and assume that $f \in M_1$ (M_1 is defined as above). Assume that there exists $u \in \overset{o}{K}$ such that $f(u) = u$ and that there exists $x_0 \in \overset{o}{K}$ such that $f'(x_0)$ is primitive. Then for every $x \in \overset{o}{K}$, there exists $\lambda(x) > 0$ such that

$$\lim_{k \to \infty} f^k(x) = \lambda(x) u , \tag{17}$$

and the map $x \to \lambda(x)$ is concave.

Proof. It is proved in Lemma 2.2 of [13] (or one can easily prove directly) that if $f \in M$ (which contains M_1) then $f'(x)$ and $f'(y)$ have the same pattern of zero and positive entries for all $x, y \in \overset{o}{K}$. In particular, $f'(x)$ is primitive for all $x \in \overset{o}{K}$, and (17) follows from Theorem 1. The concavity of $\lambda(x)$ follows from Theorem 2 and Lemma 4. []

Remark 1. If $f \in M_+$ and $f'(x_0)$ is primitive for some $x_0 \in \overset{o}{K}$, there exists $u \in \overset{o}{K}$ such that $f(u) = \lambda_0 u$ and u is unique to within scalar multiples: see Section 2 of [13] and [14]. Thus if $f \in M_+ \cap M_1$, Corollary 1 can be applied to $\lambda_0^{-1} f(x) = g(x)$. For general $f \in M$, the question of the existence of an eigenvector u in the *interior* of K appears to be subtle: see Section 3 of [13].

Corollary 2. Suppose that $n = 2m$ is an even integer, K is the standard cone in $\mathfrak{R}^n$ and $f : \overset{o}{K} \to \overset{o}{K}$ is a Borchardt map. If $u = (1, 1, \ldots, 1)$, then $f(u) = u$ and for every $x \in \overset{o}{K}$ one has

$$\lim_{k \to \infty} f^k(x) = \lambda(x) u .$$

The map $x \to \lambda(x)$ is concave. In particular, if M(a, b) denotes the AGM of positive numbers a and b, $(a, b) \to M(a, b)$ is concave.

Proof. Clearly $f \in M_1$, $f'(x_0)$ has all positive entries for every $x_0 \in \overset{o}{K}$ and $f(u) = u$, so Corollary 2 follows immediately from Corollary 1. []

The next theorem is a variant of Theorem 2.

Theorem 4. Let C, C_v and X be as in Theorem 2. Assume that $f : C_v \to C_v$ is order-preserving and that for every $x \in C_v$ there exists $u(x) \in C_v$ such that

$$\lim_{k \to \infty} f^k(x) = u(x).$$

Let D be a convex subset of X and $\psi : D \to C_v$ a homeomorphism of D onto C_v such that ψ and ψ^{-1} are both order-preserving (or both order-reversing) and $\psi^{-1} f \psi$ is convex (respectively, concave). Then the map $x \to \psi^{-1}(u(\psi(x)))$ is convex (respectively, concave) and order-preserving.

Proof. By assumption $\psi^{-1} f \psi$ is convex and order-preserving. Lemma 1 implies that $(\psi^{-1} f \psi)^k = \psi^{-1} f^k \psi$ is convex and order-preserving. The continuity of ψ and the displayed equation above then imply that $\lim_{k \to \infty} (\psi^{-1} f \psi)^k (x) = \psi^{-1}(u(\psi(x)))$. Lemma 2 implies that $\psi^{-1} u \psi$ is convex and order-preserving. []

Remark 2. In general, the fact that a homeomorphism ψ is order-preserving does not imply that ψ^{-1} is order-preserving. For example, if K is the cone of positive semidefinite self-adjoint operators on a Hilbert space H, the map $\psi(A) = A^{1/2}$ is an order-preserving homeomorphism of K onto K, but $\psi^{-1}(A) = A^2$ is not order-preserving.

If K is the standard cone in $\Re^n$, define a homeomorphism ψ of $\Re^n$ onto $\overset{o}{K}$ by

$$\psi(y) \equiv e^y = \left(e^{y_1}, e^{y_2}, \ldots, e^{y_n}\right), \tag{18}$$

so

$$\psi^{-1}(x) \equiv \log(x) = (\log(x_1), \log(x_2), \ldots, \log(x_n)).$$

Both ψ and ψ^{-1} are order-preserving maps with respect to the partial ordering induced by K.

Lemma 5. If K is the standard cone in $\Re^n$, $f \in M_+$ and $\psi(y) = e^y$ is defined by (18), then $\psi^{-1} f \psi$ is a convex, order-preserving map of $\Re^n$ to $\Re^n$.

Proof. Suppose that Γ is a finite collection of ordered pairs (r, σ), where r is a nonnegative real number and $\sigma \in K$ is a probability vector. For each $(r, \sigma) \in \Gamma$, let $c_{r\sigma}$ be a positive real and define $g : \overset{o}{K} \to \Re$ by $g(x) = \sum_{(r,\sigma)\in\Gamma} c_{r\sigma} M_{r\sigma}(x)$. To see that $y \to \log(g(e^y))$ is convex, note that if $u, v \in \Re^n$, $0 < t < 1$ and $r \geq 0$, then

$$M_{r\sigma}(e^{(1-t)u+tv}) \leq (M_{r\sigma}(e^u))^{1-t}(M_{r\sigma}(e^v))^t. \tag{19}$$

If $r = 0$, inequality (19) becomes an equality, so assume $r > 0$. Hölder's inequality gives

$$\sum_{i=1}^n \sigma_i e^{r(1-t)u_i + rtv_i} = \sum_{i=1}^n \left(\sigma_i e^{ru_i}\right)^{1-t}\left(\sigma_i e^{rv_i}\right)^t \leq \left(\sum_{i=1}^n \sigma_i e^{ru_i}\right)^{1-t}\left(\sum_{i=1}^n \sigma_i e^{rv_i}\right)^t, \tag{20}$$

and inequality (19) follows from inequality (20) by taking rth roots.

For notational convenience, if x and y are vectors in $\overset{o}{K}$ and α and β are real numbers define

$$x^\alpha y^\beta = (x_1^\alpha y_1^\beta,\ x_2^\alpha y_2^\beta, \ldots,\ x_n^\alpha y_n^\beta).$$

Proving the convexity of $\log(g(e^y))$ is then equivalent to proving

$$g(e^{(1-t)u+tv}) \leq (g(e^u))^{1-t}(g(e^v))^t \tag{21}$$

where u, v and t are as above. By virtue of inequality (19), inequality (21) follows from

$$\sum_{(r,\sigma)\in\Gamma} (c_{r\sigma}M_{r\sigma}(e^u))^{1-t}(c_{r\sigma}M_{r\sigma}(e^v))^t \leq (g(e^u))^{1-t}(g(e^v))^{1-t}. \tag{22}$$

Inequality (22) is a consequence of Hölder's inequality.

By using the above result about g, we immediately see that $\psi^{-1}f\psi$ is convex if $f \in M_+$. Of course, it is trivial that $\psi^{-1}f\psi$ is order-preserving if $f \in M_+$, because ψ^{-1}, f and ψ are order-preserving.

To complete the proof, let A denote the set of maps $f : \overset{o}{K} \to \overset{o}{K}$ such that f is order-preserving and $\psi^{-1}f\psi$ is convex. We know that $A \supset M_+$, so if we can prove that A is closed under composition and addition, it will follow that $A \supset M_+$. Closure under composition is immediate from Lemma 1. Closure under addition follows because (as is well-known) the sum of log convex functions is log convex. []

Corollary 3. Let K denote the standard cone in $\Re^n$ and suppose that $f \in M_+$ (where M_+ is defined as above). Assume that there exists $u \in \overset{o}{K}$ such that $f(u) = u$ and that $f'(x_0)$ is primitive for some $x_0 \in \overset{o}{K}$. Then for every $x \in \overset{o}{K}$ there exists $\lambda(x) > 0$ such that $\lim_{k \to \infty} f^k(x) = \lambda(x)u$, and for $y \in \Re^n$, the map $y \to \log(\lambda(e^y))$ is convex.

Proof. The first part of Corollary 3 is immediate from Theorem 1, and the convexity of $\log(\lambda(e^y))$ follows from Theorem 4 and Lemma 5. []

As a special case of Corollary 3 we obtain:

Corollary 4. Let the assumptions and the notation be as in Corollary 2. Then the mapy $\to \log(\lambda(e^y))$ is a convex map from $\Re^n$ to $\Re$. In particular, if M(a, b) denotes the AGM,$(\alpha, \beta) \to \log M(e^\alpha, e^\beta)$ is convex.

Remark 3. If I(a, b) is the integral given in (4), Corollary 2 and (3) imply that (a, b) $\to$ I(a, b) is convex and Corollary 4 implies that $(\alpha, \beta) \to \log I(e^\alpha, e^\beta)$ is concave. Given that one knows the relationship between I(a, b) and M(a, b), a specialization of the argument given here seems the easiest way to prove these convexity properties of I.

An open problem is to develop analogues of known results about the AGM of pairs of nonzero complex numbers (not merely positive real numbers) [see 7] for the new "Borchardt maps" defined here, when these maps operate on even numbers of nonzero complex numbers.

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