Perturbation Theory of Completely Mixed Bimatrix Games

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

Submitted by Richard A. Brualdi

ABSTRACT

A two-person non-zero-sum bimatrix game (A, B) is defined to be completely mixed if every solution gives a positive probability to each pure strategy of each player. Such a game is defined to be nonsingular if both payoff matrices are nonsingular. Suppose that A is perturbed to $A + \alpha G$ and B is perturbed to $B + \alpha H$, where G and H are matrices of the same size as A and B, and α is a small real number, i.e., suppose that multiple elements of each payoff matrix are perturbed simultaneously. We calculate the effect of such perturbations on the solution and values of the game for each player. When a player's payoff matrix is an M-matrix and a single diagonal element of the payoff matrix is perturbed, then that player's value is a concave function of the perturbation. A new class of completely mixed bimatrix games is analyzed.

1. INTRODUCTION

This paper develops the perturbation theory of finite, two-person, non-zero-sum games, or bimatrix games. Perturbation theory describes how small variations in the values of the parameters of payoff functions of a game affect the solutions and values of the game.

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To our knowledge, the perturbation theory of bimatrix games has not been studied before, except in the special case of zero-sum games. Studies of the perturbation theory of zero-sum matrix games began with Gross (1954) and are reviewed by Cohen (1986), who, however, overlooked the work of Mills (1956). Gal (1984) reviews the closely related perturbation theory of linear programs but does not discuss the connection with matrix games (see Mills 1956).

The perturbation theory of games in general, and of bimatrix games in particular, is of practical interest for both estimation and control. When payoff functions are estimated from data, the first derivative of the value with respect to parameters of the payoff functions indicates its sensitivity to errors in the values of those parameters, and therefore indicates which parameters should be estimated with greatest precision. Kuhn and Tucker (1956, p. viii) recognized the importance of perturbation theory for the control of games: "This study [Mills 1956] promises practical application whenever these parameters [the elements of the payoff matrix] can be controlled or altered since it indicates which changes will have a beneficial effect on the value."

A bimatrix game with m pure strategies for player 1 and n pure strategies for player 2, where $1 \leqslant m, n < \infty$, is specified by two real $m \times n$ matrices A and B. If player 1 chooses pure strategy i and player 2 chooses pure strategy j, the payoffs of the bimatrix game (A, B) to players 1 and 2 are a_{ij} and b_{ij}, respectively, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Let

$$P_n = \{ x \in R^n : x_i \geqslant 0,\ i = 1, 2, \ldots, n, \text{ and } \textstyle\sum_{i=1}^n x_i = 1 \}$$

and $P_n^+ = \{ x \in P_n : x_i > 0,\ i = 1, \ldots, n \}$. Vectors are assumed to be column vectors, and ^{T} denotes transpose. The vectors in P_n are called mixed strategies. A pair (x, y), where $x \in P_m$ and $y \in P_n$, is defined to be a *solution* of the game specified by (A, B) if and only if

$$\text{for all} \quad \xi \in P_m, \qquad \xi^T A y \leqslant x^T A y$$

and

$$\text{for all} \quad \eta \in P_n, \qquad x^T B \eta \leqslant x^T B y.$$

Given a solution (x, y), the *value* of the game for player i, $i = 1, 2$, is defined by

$$v_1(x, y) = x^T A y,$$

$$v_2(x, y) = x^T B y.$$

Define the bimatrix game (A, B) to be nonsingular if A and B are both nonsingular.

A bimatrix game is defined to be *completely mixed* when every solution (x, y) has all positive elements, i.e., $x \in P_m^+$, $y \in P_n^+$ (Jansen 1981, p. 535). This paper describes how a solution and the corresponding values of a nonsingular, completely mixed bimatrix game are affected by small perturbations of A and B.

Completely mixed bimatrix non-zero-sum games exist. For example, if

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then the set of solutions is

$$\left\{ (x, y) : x^T = \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \text{ and } y^T = \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \right\}.$$

This game is a special case for $n = 2$ of the "diagonal-cyclic" game analyzed in Section 3.

2. GENERAL RESULTS

THEOREM 1. *Suppose that the bimatrix game (A, B) is nonsingular and completely mixed (cm). Then $m = n$, and the game has a unique solution and nonzero values. The solution and values of the game for each player are given by*

$$x^T = \frac{\mathbf{1}^T B^{-1}}{\mathbf{1}^T B^{-1} \mathbf{1}}, \qquad y = \frac{A^{-1}\mathbf{1}}{\mathbf{1}^T A^{-1}\mathbf{1}},$$

$$v_1 = v_1(x, y) = \frac{1}{\mathbf{1}^T A^{-1}\mathbf{1}}, \qquad v_2 = v_2(x, y) = \frac{1}{\mathbf{1}^T B^{-1}\mathbf{1}},$$

where $\mathbf{1}$ is the column vector of length n with every element equal to 1.

This result of Raghavan and Heuer is stated and proved by Jansen (1981, Theorem 3.12, p. 536).

While it is not surprising that A uniquely determines v_1 and B uniquely determines v_2, it is somewhat surprising that B uniquely determines x, player 1's optimal mixed strategy, while A uniquely determines y.

For a pair (A, B) of real matrices that satisfies the hypotheses of Theorem 1, it follows that

$$\frac{\mathbf{1}^T B^{-1}}{\mathbf{1}^T B^{-1}\mathbf{1}} > \mathbf{0}^T, \qquad \frac{A^{-1}\mathbf{1}}{\mathbf{1}^T A^{-1}\mathbf{1}} > \mathbf{0},$$

where $\mathbf{0}$ is the column vector with all n elements 0, and the inequalities $>$ apply element by element. For a zero-sum game, these inequalities imply conversely that the game is completely mixed. However, these inequalities do not guarantee that the game (A, B) with nonzero sum is completely mixed. For example, if $A = B = I_2$, where I_n is the $n \times n$ identity matrix, then A and B satisfy the previous inequalities, but (A, B) is not cm, because one solution is

$$x = y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

For the remainder of this section, let A, B, G, and H be fixed $n \times n$ real matrices, and for each real number α, define

$$L = L(\alpha) = A + \alpha G, \qquad M = M(\alpha) = B + \alpha H.$$

LEMMA 1. *If (A, B) is a nonsingular, completely mixed bimatrix game, then there exists a real number $r > 0$ such that for all real α with $|\alpha| < r$, $(L(\alpha), M(\alpha))$ is a nonsingular, completely mixed bimatrix game.*

Proof. Jansen (1981, Theorem 3.15, p. 536) proved that the class of all completely mixed bimatrix games is open in the set of pairs of $m \times n$ real matrices. It is also known that the set of nonsingular matrices is open. ∎

LEMMA 2. *The following derivatives exist and are equal to the formulas given:*

$$\frac{dL(\alpha)}{d\alpha} = G, \qquad \frac{dM(\alpha)}{d\alpha} = H,$$

and when $L^{-1}(\alpha)$ and $M^{-1}(\alpha)$, respectively, exist,

$$\frac{dL^{-1}(\alpha)}{d\alpha} = -L^{-1}(\alpha) G L^{-1}(\alpha), \qquad \frac{dM^{-1}(\alpha)}{d\alpha} = -M^{-1}(\alpha) H M^{-1}(\alpha).$$

When $\alpha = 0$,

$$\frac{dL^{-1}(0)}{d\alpha} = -A^{-1}GA^{-1}, \qquad \frac{dM^{-1}(0)}{d\alpha} = -B^{-1}HB^{-1}.$$

Proof. The existence of the derivatives follows from Lemma 1. The formulas for $dL/d\alpha$ and $dM/d\alpha$ are immediate. When L^{-1} exists, $LL^{-1} = I_n$, so by the chain rule, $(dL/d\alpha)L^{-1} + L(dL^{-1}/d\alpha) = 0$. Hence, $dL^{-1}/d\alpha = -L^{-1}GL^{-1}$. The computation of $dM^{-1}/d\alpha$ is the same. ■

Given the nonsingular, completely mixed bimatrix games (A, B) and $(L(\alpha), M(\alpha))$, $L(\alpha) = A + \alpha G$, $M(\alpha) = B + \alpha H$, for real α such that $|\alpha| < r$ where r is given by Lemma 1, define for $|\alpha| < r$

$$x^T(\alpha) = \frac{\mathbf{1}^T M^{-1}(\alpha)}{\mathbf{1}^T M^{-1}(\alpha)\mathbf{1}}, \qquad y(\alpha) = \frac{L^{-1}(\alpha)\mathbf{1}}{\mathbf{1}^T L^{-1}(\alpha)\mathbf{1}},$$

$$v_1(\alpha) = \frac{1}{\mathbf{1}^T L^{-1}(\alpha)\mathbf{1}}, \qquad v_2(\alpha) = \frac{1}{\mathbf{1}^T M^{-1}(\alpha)\mathbf{1}}.$$

Thus, Theorem 1 applied to the nonsingular, completely mixed game $(L(\alpha), M(\alpha))$ implies that $(x(\alpha), y(\alpha))$ is the unique solution of $(L(\alpha), M(\alpha))$ and $v_i(\alpha)$ is the value of $(L(\alpha), M(\alpha))$ to player i, $i = 1, 2$.

THEOREM 2. *If* (A, B) *is a nonsingular, completely mixed bimatrix game,* G *and* H *are* $n \times n$ *real matrices, and* α *is real, then*

$$\frac{dx^T(0)}{d\alpha} = \frac{(\mathbf{1}^T B^{-1}HB^{-1}\mathbf{1})\mathbf{1}^T B^{-1}}{(\mathbf{1}^T B^{-1}\mathbf{1})^2} - \frac{\mathbf{1}^T B^{-1}HB^{-1}}{\mathbf{1}^T B^{-1}\mathbf{1}},$$

$$\frac{dy(0)}{d\alpha} = \frac{(\mathbf{1}^T A^{-1}GA^{-1}\mathbf{1})A^{-1}\mathbf{1}}{(\mathbf{1}^T A^{-1}\mathbf{1})^2} - \frac{A^{-1}GA^{-1}\mathbf{1}}{\mathbf{1}^T A^{-1}\mathbf{1}},$$

$$\frac{dv_1(0)}{d\alpha} = \frac{\mathbf{1}^T A^{-1}GA^{-1}\mathbf{1}}{(\mathbf{1}^T A^{-1}\mathbf{1})^2}, \qquad \frac{dv_2(0)}{d\alpha} = \frac{\mathbf{1}^T B^{-1}HB^{-1}\mathbf{1}}{(\mathbf{1}^T B^{-1}\mathbf{1})^2},$$

$$\frac{d^2v_1(0)}{d\alpha^2} = 2\left[\frac{(\mathbf{1}^T A^{-1}GA^{-1}\mathbf{1})^2}{(\mathbf{1}^T A^{-1}\mathbf{1})^3} - \frac{(\mathbf{1}^T A^{-1}GA^{-1}GA^{-1}\mathbf{1})}{(\mathbf{1}^T A^{-1}\mathbf{1})^2}\right],$$

and the formula for $d^2v_2(0)/d\alpha^2$ *has the same form as the formula for* $d^2v_1(0)/d\alpha^2$ *with* A *replaced by* B *and* G *replaced by* H.

Proof. The formulas follow by the formulas of Theorem 1 applied to the game $(L(\alpha), M(\alpha))$, repeated use of the chain rule, and the formulas of Lemma 2. ■

As the solutions $(x(\alpha), y(\alpha))$ must be pairs of probability vectors, the sums of the derivatives of the elements of $x(\alpha)$ and $y(\alpha)$ should be 0. It is easy to check that the formulas of Theorem 2 satisfy $[dx^T(0)/d\alpha]\mathbf{1} = 0$ and $\mathbf{1}^T[dy(0)/d\alpha] = 0$, as desired.

If $v_1 > 0$, then

$$\operatorname{sign}\left(\frac{d^2v_1(0)}{d\alpha^2}\right) = \operatorname{sign}\left\{(\mathbf{1}^TA^{-1}GA^{-1}\mathbf{1})^2 - (\mathbf{1}^TA^{-1}\mathbf{1})\left[\mathbf{1}^TA^{-1}(GA^{-1})^2\mathbf{1}\right]\right\}.$$

It would be desirable to find general conditions on A and G which determine the sign on the right.

A useful special case arises when A and G are both diagonal matrices and all diagonal elements of A are positive. Then the matrices $F = GA^{-1}$ and $D = A^{-1}$ are also diagonal matrices and all diagonal elements of D are positive. If $F = \operatorname{diag}(f_i)$, then

$$(\mathbf{1}^TA^{-1}GA^{-1}\mathbf{1})^2 - (\mathbf{1}^TA^{-1}\mathbf{1})\left[\mathbf{1}^TA^{-1}(GA^{-1})^2\mathbf{1}\right]$$

$$= \left(\sum_{i=1}^{n} d_i f_i\right)^2 - \left(\sum_{i=1}^{n} d_i\right)\left(\sum_{i=1}^{n} d_i f_i^2\right) \leqslant 0,$$

where the inequality follows from the Cauchy-Schwarz inequality This inequality applies to the diagonal-cyclic game considered in the next section. The referee generously pointed out that, more generally, if A and G are both symmetric and A is positive definite, then by the Cauchy-Schwarz inequality

$$(\mathbf{1}^TA^{-1}GA^{-1}\mathbf{1})^2 - (\mathbf{1}^TA^{-1}\mathbf{1})\left[\mathbf{1}^TA^{-1}(GA^{-1})^2\mathbf{1}\right] \leqslant 0.$$

3. SPECIAL CASES

Zero-Sum Case

When the bimatrix game has sum zero, then $B = -A$, $G = -H$, $x^T = x^T(0) = \mathbf{1}^TA^{-1}/\mathbf{1}^TA^{-1}\mathbf{1}$, and $y = y(0) = A^{-1}\mathbf{1}/\mathbf{1}^TA^{-1}\mathbf{1}$. Then Theorem 2 asserts that $dv_1(0)/d\alpha = -dv_2(0)/d\alpha = x^TGy$. This formula is the special case when A is completely mixed of Theorem 1 of Mills (1956, pp. 184–185).

Perturbation of a Single Element

Theorem 2 also describes as a special case the perturbation of a single element a_{ij} of the payoff matrix A of a zero-sum two-person completely mixed matrix game (Cohen 1986). Let E_{ij} denote the $n \times n$ matrix with all elements equal to 0 except for the (i, j) element, which equals 1. If e_i is the column vector with ith element equal to 1 and all other elements equal to 0, then $E_{ij} = e_i e_j^T$. Replacing G by E_{ij} gives, for example, from Theorem 2,

$$\frac{d^2 v_1}{da_{ij}^2} = 2\left(\frac{1}{\mathbf{1}^T A^{-1}\mathbf{1}}\right)\left(\frac{i\text{th column sum of } A^{-1}}{\mathbf{1}^T A^{-1}\mathbf{1}}\right) \times \left(\frac{j\text{th row sum of } A^{-1}}{\mathbf{1}^T A^{-1}\mathbf{1}}\right)\phi_{ji}(A),$$

where

$$\phi_{ji}(A) = (j\text{th row sum of } A^{-1})(i\text{th column sum of } A^{-1}) - (A^{-1})_{ji}(\mathbf{1}^T A^{-1}\mathbf{1}).$$

The factor $1/(\mathbf{1}^T A^{-1}\mathbf{1})$ corresponds to $v_1(A)$, and in a zero-sum game, the succeeding factors correspond to x_i, y_j, and $\phi_{ji}(A)$ as in the formula (2.5) of Cohen (1986, p. 158).

Diagonal-Cyclic Case

Fix $a_i > 0$ and $b_i > 0$ for $i = 1, \ldots, n$.

THEOREM 3. *If*

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_n \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & b_1 & 0 & \cdots & 0 \\ 0 & 0 & b_2 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & b_{n-1} \\ b_n & 0 & 0 & \cdots & 0 \end{pmatrix},$$

then the bimatrix game (A, B) is nonsingular and completely mixed. The *game* (A, B) *has a unique solution* (x, y) *with*

$$x_i = v_2/b_i, \quad y_i = v_1/a_i, \qquad i = 1,\dots,n,$$

where

$$v_1 = \left(\sum_{i=1}^{n} \frac{1}{a_i}\right)^{-1}, \qquad v_2 = \left(\sum_{i=1}^{n} \frac{1}{b_i}\right)^{-1}.$$

Proof. Clearly A and B are nonsingular. We shall first show that there exists a solution (x, y) with all positive elements, and then that every solution has all positive elements. It is immediate that

$$A^{-1} = \begin{pmatrix} 1/a_1 & 0 & \cdots & 0 \\ 0 & 1/a_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 1/a_n \end{pmatrix},$$

$$B^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1/b_n \\ 1/b_1 & 0 & \cdots & 0 & 0 \\ 0 & 1/b_2 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & 1/b_{n-1} & 0 \end{pmatrix}.$$

Then, as in Theorem 1,

$$v_1 = \left(\sum_{r=1}^{n} \frac{1}{a_r}\right)^{-1}, \qquad v_2 = \left(\sum_{r=1}^{n} \frac{1}{b_r}\right)^{-1}.$$

A solution (x, y) with all positive elements is

$$x_i = v_2/b_i, \quad y_i = v_1/a_i, \qquad i = 1,\dots,n,$$

and no other solution with all positive elements is possible.

To prove that every solution has all elements positive, let (x, y) be any solution. For any ξ in P_n, define the support of ξ by $S(\xi) = \{i : \xi_i > 0\}$. Then since

$$x^T A y = \sum_{i \in S(y)} a_i x_i y_i \geqslant \xi^T A y \qquad \text{for all} \quad \xi \in P_n,$$

player 1 cannot maximize his payoff if he assigns any probability to pure strategies i that are not in $S(y)$; hence $S(x) \subseteq S(y)$. Similarly, if we interpret subscripts modulo n so that y_{n+1} means y_1,

$$x^T B y = \sum_{i \in S(x)} b_i x_i y_{i+1} \geqslant x^T B \eta \qquad \text{for all} \quad \eta \in P_n;$$

hence if $i + 1 \pmod n \in S(y)$, then $i \in S(x)$. Since $S(x) \subseteq S(y)$, $i + 1 \pmod n \in S(y)$ implies $i \in S(y)$. Because $S(y)$ cannot be empty, the only possibility is that $S(y) = \{1, \dots, n\}$. A similar argument gives also $S(x) = \{1, \dots, n\}$. Thus (A, B) is nonsingular and completely mixed. ■

Because the solution and values of (A, B) are known explicitly, the perturbation theory of the game (A, B) can also be carried out explicitly in certain cases. We consider first $G = E_{ii} = e_i e_i^T$ for some fixed i, $1 \leqslant i \leqslant n$, and then $G = \operatorname{diag}(g_i)$, which is the diagonal matrix with $g_{ii} = g_i$.

If $G = E_{ii} = e_i e_i^T$, then

$$\frac{dv_1}{d\alpha} = \frac{dv_1}{da_{ii}} = \frac{(1/a_i)^2}{(\Sigma_r 1/a_r)^2} = y_i^2 > 0.$$

Hence

$$0 < \frac{dv_1}{da_{ii}} < 1 \quad \text{and,} \quad \text{if } n > 1, \quad 0 < \sum_{r=1}^{n} \frac{dv_1}{da_{rr}} < 1.$$

Differentiating again,

$$\frac{d^2 v_1}{d\alpha^2} = \frac{d^2 v_1}{da_{ii}^2} = -\frac{\dfrac{2}{a_i^3} \displaystyle\sum_{r \neq i} \frac{1}{a_r}}{\left(\displaystyle\sum_{r=1}^{n} \frac{1}{a_r} \right)^3}.$$

Hence $d^2v_1/da_{ii}^2 < 0$ if $n > 1$. In words, the value for player 1 is a strictly increasing, strictly concave (if $n > 1$) function of the ith element of A. Concavity has the economic interpretation of decreasing returns: equal marginal increments in the ith element of A produce successively smaller increments in the value of player 1. Finally,

$$\frac{dy_j}{da_{ii}} = \begin{cases} \dfrac{y_i^2}{a_j} > 0 & \text{if} \quad j \neq i, \\ \dfrac{y_i(y_i - 1)}{a_i} \leqslant 0 & \text{if} \quad j = i. \end{cases}$$

Hence $dy_i/da_{ii} < 0$ if $n > 1$. As expected, $\Sigma_{j=1}^n dy_j/da_{ii} = 0$. Exactly symmetrical formulas hold for dv_2/db_i, d^2v_2/db_i^2, and dx_j/db_i.

Now if $G = \text{diag}(g_i)$, then by Theorem 2,

$$\frac{d^2v_1}{d\alpha^2} = 2\left(\sum_r \frac{1}{a_r}\right)^{-3}\left[\left(\sum_r \frac{g_r}{a_r^2}\right)^2 - \left(\sum_s \frac{1}{a_s}\right)\left(\sum_r \frac{g_r^2}{a_r^3}\right)\right].$$

By the Cauchy-Schwarz inequality, in the form used at the end of Section 2, $d^2v_1/d\alpha^2 \leqslant 0$. Similarly, using Theorem 2 again,

$$\frac{dy_j}{d\alpha} = \frac{1/a_j}{(\Sigma_r 1/a_r)^2} \sum_r \frac{1}{a_r}\left(\frac{g_r}{a_r} - \frac{g_j}{a_j}\right).$$

Thus, $dy_j/d\alpha > 0$ if $g_j/a_j = \min_r g_r/a_r$ and for some $s \neq j$, $g_s/a_s > g_j/a_j$; $dy_j/d\alpha < 0$ if $g_j/a_j = \max_r g_r/a_r$ and for some $s \neq j$, $g_s/a_s < g_j/a_j$; and $dy_j/d\alpha = 0$ if g_j/a_j is a constant independent of j.

Analogous formulas may be derived for perturbations of the matrix B by the matrix H provided H has positive elements in exactly those positions where B has positive elements.

M-Matrix Case

A real $n \times n$ matrix A is called an M-matrix if $A = sI_n - M$, where M has nonnegative elements and s is at least as big as the Perron-Frobenius root of M. If A is a nonsingular M-matrix, then A is completely mixed (Karlin 1959, p. 52) as the payoff matrix of a zero-sum game; moreover, $\phi_{ii}(A) < 0$, $i = 1, \ldots, n$ (Cohen 1986).

THEOREM 4. *Let (A, B) be a nonsingular, completely mixed bimatrix game. If A is also an M-matrix, then for $i, j = 1, \ldots, n$,*

$$\frac{dy_j}{da_{ji}} < 0, \qquad \frac{d^2 v_1}{da_{ii}^2} < 0.$$

If B is also an M-matrix, then for $i, j = 1, \ldots, n$,

$$\frac{dx_j}{da_{ij}} < 0, \qquad \frac{d^2 v_2}{db_{ii}^2} < 0.$$

Proof. In the formula of Theorem 2 for $dy(0)/d\alpha$, replace G by E_{ij} to get dy/da_{ij}. Using Theorem 1 gives easily $dy_k/da_{ij} = y_j v_1 \phi_{ki}(A)$ as in Theorem 2 of Cohen (1986, p. 156). Then $dy_i/da_{ij} = y_j v_1 \phi_{ii}(A) < 0$ because $y_j > 0$, $v_1 > 0$ and $\phi_{ii}(A) < 0$. That $d^2v_1/da_{ii}^2 < 0$ follows from the more general formula above for d^2v_1/da_{ij}^2 and the fact that $\phi_{ii}(A) < 0$. Similar calculations give the other two claimed inequalities. ■

It is easy to obtain completely explicit formulas for the expressions in Theorem 4 when A is diagonal, $B = -A$.

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