Spectral Inequalities for Matrix Exponentials

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ABSTRACT

This note generalizes an inequality of Bernstein as follows. If C is an $n \times n$ complex matrix and $C^{(k)}$ is the kth compound of C, $1 \leqslant k \leqslant n$, $N = \binom{n}{k}$, and if the eigenvalues of $C^{(k)}$ are labeled in order of decreasing magnitude $|\lambda_1(C^{(k)})| \geqslant |\lambda_2(C^{(k)})| \geqslant \cdots \geqslant |\lambda_N(C^{(k)})|$, define the partial trace $\mathrm{tr}_i^{(k)}(C)$ by

$$\mathrm{tr}_i^{(k)}(C) = \sum_{h=1}^{i} \lambda_h(C^{(k)}), \qquad i = 1, \ldots, N.$$

Then for any complex $n \times n$ matrix A,

$$\mathrm{tr}_i^{(k)}(e^A e^{A^*}) \leqslant \mathrm{tr}_i^{(k)}(e^{A+A^*}), \qquad i = 1, \ldots, N,$$

with equality if A is normal or $k = n$. A spectral inequality of K. Fan is also generalized through the use of compound matrices.

1. INTRODUCTION

Mathematical models in control theory [1], statistical mechanics [5], and population biology [2] lead to formulas containing $e^A e^B$ and e^{A+B}, for noncommuting $n \times n$ matrices A and B. The behaviors of these models depend on functions of the eigenvalues of $e^A e^B$ and e^{A+B}. The purpose of

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this note is to extend a recent inequality that compares the eigenvalues of $e^A e^B$ with those of e^{A+B} in the special case when $B = A^*$.

Bernstein [1] proved, among other inequalities, that if A is a real $n \times n$ matrix, $1 < n < \infty$, A^T is the transpose of A, and $\operatorname{tr}(A)$ is the trace of A, then

$$\operatorname{tr}(e^A e^{A^T}) \leqslant \operatorname{tr}(e^{A+A^T}). \tag{1.1}$$

Bernstein's proof of (1.1) relies on Theorem 3 of Fan [3, p. 654]. This note generalizes Fan's theorem and then exploits that generalization fully to extend (1.1). The remainder of this introductory section gives some notation and definitions.

As usual, for any complex $n \times n$ matrix C, let C^* denote the conjugate transpose of C. A complex matrix C is normal if $CC^* = C^*C$. The kth compound $C^{(k)}$ of C, for $k = 1, \dots, n$, is the $N \times N$ matrix, where $N = \binom{n}{k}$, the elements of which are the determinants of all the possible $k \times k$ submatrices of C that consist of the intersections of rows $i_1, i_2, \dots, i_k$, where $1 \leqslant i_1 < \cdots < i_k \leqslant n$, and of columns $j_1, j_2, \dots, j_k$, where $1 \leqslant j_1 < \cdots < j_k \leqslant n$. The elements of $C^{(k)}$ are ordered lexicographically by the indices of the rows or columns of C that are included. (See [4] for a review of compound matrices.) A first key fact (e.g., [4]) is the Binet-Cauchy formula: for any complex $n \times n$ matrices A and B, $A^{(k)}B^{(k)} = (AB)^{(k)}$, $k = 1, \dots, n$. A second key fact is that if $\lambda_i(C)$, $i = 1, \dots, n$, are the eigenvalues of C (some of which may be repeated), then the N eigenvalues of $C^{(k)}$ are all the products of eigenvalues of C taken k at a time:

$$\lambda_{i_1}(C)\lambda_{i_2}(C) \cdots \lambda_{i_k}(C), \qquad \text{for} \quad 1 \leqslant i_1 < \cdots < i_k \leqslant n.$$

To illustrate, $C^{(1)} = C$ and $C^{(n)} = \det C$, where det = determinant.

Assuming the eigenvalues of C are labeled in order of decreasing magnitude $|\lambda_1(C)| \geqslant |\lambda_2(C)| \geqslant \cdots \geqslant |\lambda_n(C)|$, define the partial trace $\operatorname{tr}_i^{(k)}(C)$ by

$$\operatorname{tr}_i^{(k)}(C) = \sum_{h=1}^{i} \lambda_h(C^{(k)}), \qquad i = 1, \dots, N = \binom{n}{k}. \tag{1.2}$$

Thus $\operatorname{tr}_i^{(k)}(C) = \operatorname{tr}_i^{(1)}(C^{(k)})$. To illustrate, $\operatorname{tr}_N^{(k)}(C)$ is the kth elementary symmetric function of the eigenvalues of C; in particular, $\operatorname{tr}_n^{(1)}(C)$ is the usual trace of C, and $\operatorname{tr}_1^{(1)}(C)$ is the spectral radius of C. When C is nonnegative definite, ordering the eigenvalues of C by decreasing magnitude amounts to ordering them by the usual order on nonnegative real numbers; thus $\operatorname{tr}_1^{(k)}(C)$ is the product of the k biggest eigenvalues of C.

2. INEQUALITIES FOR EXPONENTIALS OF A AND A^*

THEOREM 1. *For any complex $n\times n$ matrix C and for any positive integer r,*

$$\operatorname{tr}_i^{(k)}[C^r(C^r)^*] \leqslant \operatorname{tr}_i^{(k)}\left[(CC^*)^r\right], \qquad k=1,\dots,n, \quad i=1,\dots,\binom{n}{k}, \quad (2.1)$$

with equality if C is normal or $k=n$.

Proof. The arguments of $\operatorname{tr}_i^{(k)}(\cdot)$ in (2.1) are Hermitian nonnegative definite and therefore have real nonnegative eigenvalues, so the relation $\leqslant$ in (2.1) is defined.

Fan [3, p. 654] proved that for any complex $n\times n$ matrix C and for any positive integer r,

$$\operatorname{tr}_i^{(1)}[C^r(C^r)^*] \leqslant \operatorname{tr}_i^{(1)}\left[(CC^*)^r\right], \qquad i=1,\dots,n. \qquad (2.2)$$

Now if C is replaced by $C^{(k)}$, then (by the Binet-Cauchy formula) $(C^{(k)})^r = (C^r)^{(k)}$ and $(C^r)^{(k)*}=[(C^r)^*]^{(k)}$, so the argument on the left of (2.2) becomes $[C^r(C^r)^*]^{(k)}$, and by the definition (1.2) we have $\operatorname{tr}_i^{(1)}([C^r(C^r)^*]^{(k)}) = \operatorname{tr}_i^{(k)}[C^r(C^r)^*]$. Similarly, replacing C by $C^{(k)}$ in the argument on the right of (2.2) and using the Binet-Cauchy formula give $\operatorname{tr}_i^{(1)}[(C^{(k)}C^{(k)*})^r] = \operatorname{tr}_i^{(k)}[(CC^*)^r]$.

If C is normal, then $C^r(C^r)^*=(CC^*)^r$, so equality holds in (2.1). If $k=n$, both sides of (2.1) equal $(\det C)^r(\det C^*)^r$. ■

THEOREM 2. *For any complex $n\times n$ matrix A,*

$$\operatorname{tr}_i^{(k)}(e^{A}e^{A^*}) \leqslant \operatorname{tr}_i^{(k)}(e^{A+A^*}), \qquad k=1,\dots,n, \quad i=1,\dots,\binom{n}{k}, \quad (2.3)$$

with equality if A is normal or $k=n$.

Proof. In (2.1), let $C = e^{A/r}$. Then, since $(e^A)^* = e^{A^*}$,

$$\mathrm{tr}_i^{(k)}(e^A e^{A^*}) \leqslant \mathrm{tr}_i^{(k)}\left[(e^{A/r}e^{A^*/r})^r\right]. \tag{2.4}$$

Let $r \uparrow \infty$ in (2.4). By the exponential product formula of Sophus Lie (e.g., [6]), $(e^{A/r}e^{A^*/r})^r \to e^{A+A^*}$, which implies (2.3).

Equality holds in (2.3) when A is normal because then e^A is normal. ∎

It would be interesting to know necessary and sufficient conditions for equality in (2.3).

The special case of Theorem 2 when A is real, $k = 1$ and $i = n$ is (1.1) above, first proved in [1].

Dennis S. Bernstein (personal communication, 1 June 1988) points out that the square root of both sides of (2.3) in the special case $i = k = 1$ yields another known inequality: $\|e^{Ax}\| \leqslant e^{\mu(A)x}$, where $\|\cdot\|$ is the spectral norm (the matrix norm induced by the Euclidean vector norm), x is any n-vector, and $\mu(A)$ is the logarithmic "norm" (also called the logarithmic derivative or the measure of a matrix). See e.g. Torsten Ström, On logarithmic norms, *SIAM J. Numer. Anal.* 12(5):741–753 (1975), Lemma 1c(5). Thus (2.3) unifies (1.1) with a standard inequality involving the logarithmic norm.

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