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THE SENSITIVITY OF EXPECTED SPANNING TREES IN ANISOTROPIC RANDOM GRAPHS

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Consider graphs on the vertex set $V=\{1, 2, \ldots, n\}$, $1<n<\infty$, in which the edge between vertices i and j occurs with probability $p_{ij}=p_{ji}$, $0\leqslant p_{ij}\leqslant 1$, independently for all edges. Let $P=(p_{ij})$ be the $n\times n$ symmetric matrix of edge probabilities with $p_{ii}=0$, $i=1, \ldots, n$. Let T be the random number of spanning trees. $E(T|P)$ denotes the redundancy, i.e., the expected number of spanning trees in random graphs with edge probability matrix P. An explicit (determinantal) formula for the sensitivity of the redundancy to changes in any edge probability, namely, $\partial E(T|P)/\partial p_{ij}$, shows that this sensitivity equals the redundancy of random graphs in which vertices i and j have been collapsed to a single vertex or are connected with probability 1. There is an analogous formula for directed graphs.

1. Random-edge graphs

Consider graphs (Tutte 1984) on the set V of vertices, $V=\{1, \ldots, n\}$, $1<n<\infty$. Suppose that the undirected edge $\{i, j\}$ between i and j occurs with fixed but arbitrary probability $p_{ij}=p_{ji}$, $0\leqslant p_{ij}\leqslant 1$, independently for all distinct pairs i, j that satisfy $1\leqslant i<j\leqslant n$. Define $p_{ii}=0$, $i=1, \ldots, n$. Let $P=(p_{ij})$ be the $n\times n$ symmetric matrix of edge probabilities. Random graphs or edge probability matrices P with at least two different off-diagonal edge probabilities $p_{ij}\neq p_{gh}$, $i\neq g$ or $j\neq h$, will be called *anisotropic* to distinguish them from *isotropic* random graphs in which, by definition, for some $p\in[0, 1]$, $p_{ij}=p$, for all $i\neq j$. By this definition, anisotropic random graphs necessarily have $n>3$ vertices, since the symmetry of P implies that random graphs with only two vertices are always isotropic: $p_{12}=p_{21}$.

Let T be the (random) number of spanning trees of a random graph, i.e., the number of trees that are incident to every vertex in V. Let $E(.)$ denote expected

value or mean. Thus $E(T|P)$ denotes the expected number of spanning trees in random graphs with edge probability matrix P.

Graphs with randomly deleted edges may model a communication network in which communication links are subject to random failure, as in systems of strategic command and control (Ford 1985). The expected number of spanning trees could then be interpreted as a network's redundancy, or mean number of distinct paths of communication among all verices. The number of spanning trees in a fixed graph is sometimes called the complexity of the graph, so the expected number of spanning trees in random graphs could be called the mean complexity of the random graphs.

In efforts to alter the redundancy of a network, it is obviously useful to know how the redundancy changes with small changes in the probability of survival of each edge, namely, $\partial E(T|P)/\partial p_{ij}$, $i, j=1, \ldots, n$. Buzacott (1980, p. 323) calls $\partial E(T|P)/\partial p_{ij}$ the "importance factor" of edge $\{i, j\}$ for the redundancy $E(T|P)$. The first purpose of this note (sections 2, 3) is to compute the importance factors, or sensitivities, for anisotropic random graphs (see eq. (2)). I shall also define anisotropic random directed graphs and compute (section 4) the importance factors for them (see eq. (4)).

The second purpose of this note is to prove (section 5) that the sum of the importance factors for anisotropic random graphs is larger, the more evenly the edge probabilities are distributed, in the following sense.

For any $n\times n$ real matrix A with zero diagonal ($a_{ii}=0$, $i=1, \ldots, n$), let $p(A) = \sum_{i,j} a_{ij}/[n(n-1)]$ be the average of the off-diagonal elements of A. Define $\bar{A}$ to be the equisummed matrix of such a (real, zero diagonal) matrix A if $\bar{A}$ has zero diagonal elements and all off-diagonal elements $\bar{a}_{ij}$, $i\neq j$, equal to $p(A)$. By construction, $\sum_{i,j} a_{ij}=\sum_{i,j}\bar{a}_{ij}$.

For an anisotropic matrix P of edge probabilities, $\bar{P}$ gives the edge probabilities of isotropic random graphs with the same total of edge probabilities. For $0\leqslant\alpha\leqslant 1$, define $P_\alpha=(1-\alpha)P+\alpha\bar{P}$. As α increases from 0 to 1, P_α becomes closer to the edge probabilities of isotropic random graphs, while the total of edge probabilities remains constant.

I showed previously that if P is anisotropic, then the expected number $E_\alpha(T) \equiv E(T|P_\alpha)$ of spanning trees of random graphs with edge probabilities P_α increases strictly with α in [0, 1] (Cohen 1986, Cor. 3.3). Thus, for a given sum of edge probabilities, the redundancy $E_\alpha(T)$ increases, the closer the matrix of edge probabilities is to being isotropic (as measured by increasing α in $P_\alpha=(1-\alpha)P+\alpha\bar{P}$).

Here I show that for anisotropic random graphs,

$$S(\alpha)\equiv \sum_{i<j} \partial E(T|P_\alpha)/\partial p_{ij}$$

increases strictly with α in [0, 1]. Thus for a given sum of edge probabilities, the total sensitivity increases, the closer the matrix of edge probabilities is to being isotropic as measured by α.

Finally, I conjecture (section 6) that the closer edge probabilities are to being isotropic, as measured by α in P_α, the more nearly uniform are the sensitivities after normalization by their total $S(\alpha)$. "Nearness to uniform" is made precise by a claim about majorization (Marshall and Olkin 1979), which I am able to prove only for $n=3$ vertices. Because I lack a proof for general n or a counter-example, this note is only a progress report. I would welcome a resolution of the conjecture.

2. Direct computation of importance factors

A formula (1) for $E(T|P)$ from Cohen (1986) generalizes the matrix-tree theorem attributed to Kirchhoff.

For any $n\times n$ real matrix $A=(a_{ij})$, following the terminology of Tutte (1984, p. 138), define the $n\times n$ Kirchhoff matrix $K(A)$ of A by

$$K_{ii}=\sum_{j\neq i} a_{ij},\quad i=1,\ldots,n,$$

$$K_{ij}=-a_{ij},\quad 1\leqslant i\neq j<n.$$

Let det A be the determinant of A and let $A(i_1,\ldots,i_q)$ be the $(n-q)\times(n-q)$ principal submatrix of A left after deleting rows and columns $i_1,\ldots,i_q$, for $1\leqslant q\leqslant n$. Thus det $[A(i)]$ is the determinant of the $(n-1)\times(n-1)$ matrix that remains after row and column i are deleted from A. Then (Cohen 1986)

$$E(T|P)=\det\{[K(P)](h)\},\quad h=1,\ldots,n. \tag{1}$$

Then, by the chain rule,

$$\begin{aligned}\partial E(T|P)/\partial p_{ij}=&(\partial\det\{[K(P)](h)\}/\partial\{[K(P)](h)\}_{ii})(\partial\{[K(P)](h)\}_{ii}/\partial p_{ij})\\&+(\partial\det\{[K(P)](h)\}/\partial\{[K(P)](h)\}_{ij})(\partial\{[K(P)](h)\}_{ij}/\partial p_{ij})\\&+(\partial\det\{[K(P)](h)\}/\partial\{[K(P)](h)\}_{ji})(\partial\{[K(P)](h)\}_{ji}/\partial p_{ij})\\&+(\partial\det\{[K(P)](h)\}/\partial\{[K(P)](h)\}_{jj})(\partial\{[K(P)](h)\}_{jj}/\partial p_{ij}).\end{aligned}$$

Now take $h=i$. Then row and column i of $K(P)$ are absent from $\{[K(P)(i)\}$, so the first three terms on the right vanish. Then $\partial\{[K(P)](i)\}_{jj}/\partial p_{ij}=1$ implies

$$\partial E(T|P)/\partial p_{ij}=\det\{[K(P)](i,j)\}. \tag{2}$$

Comparison of (2) with Corollary 2.1 of Cohen (1986) establishes an unanticipated identity: the sensitivity $\partial E(T|P)/\partial p_{ij}$ of the redundancy $E(T|P)$ to p_{ij} equals the redundancy of random graphs in which vertices i and j have been collapsed to a single vertex (call it i^*). (In the random graph where i and j have been collapsed to i^*, an edge $\{g,h\}$, $g\neq i^*$ and $h\neq i^*$, is assumed to occur with the probability that $\{g,h\}$ occurs in the original graph, and an edge $\{h,i^*\}$, $h\neq i^*$, occurs in the collapsed graph with the probability that either edge $\{h,i\}$ or $\{h,j\}$ occurs in the original graph; and all edges in the collapsed graph occur independently.)

3. Computation of importance factors using Buzacott's formula

Buzacott (1980, p. 323) established that, for any measure Q of reliability in which an element p_{ij} of P appears linearly, the importance factor of that element satisfies

$$p_{ij}(\partial Q/\partial p_{ij})=Q-Q_0$$

where Q_0 is the value of the reliability measure for random graphs in which p_{ij} is replaced by 0 (and, necessarily for graphs, p_{ji} is also replaced by 0) and all other elements of P remain unchanged. Essam (these Proceedings, p. 51) observed that the above formula holds if Q is the expectation of any function defined on all possible subsets of edges.

For fixed i and j, let P_0 have all elements equal to the corresponding elements of P except that $(P_0)_{ij}=(P_0)_{ji}=0$. Then, in Buzacott's formula, take

$$Q=E(T|P)=\det\{[K(P)](i)\},$$

$$Q_0=E(T|P_0)=\det\{[K(P_0)](i)\}.$$

If e_j denotes the column n-vector with all elements 0 except a 1 as the jth element, then

$$j\text{th column of } P_0=(j\text{th column of } P)-p_{ij}e_i.$$

Therefore

$$\det\{[K(P_0)](i)\}=\det\{[K(P)](i)\}-p_{ij}\det\{[K(P)](i,j)\}.$$

Hence, from Buzacott's formula

$$p_{ij}(\partial E(T|P)/\partial p_{ij}) = \det\{[K(P)](i)\} - \det\{[K(P_0)](i)\}$$
$$= p_{ij}\det\{[K(P)](i,j)\},$$

as found directly in (2).

4. Importance factors for expected spanning intrees of digraphs

Let T_1 denote the number of spanning intrees (Tutte 1984, Cohen 1986) to vertex 1 in random digraphs in which p_{ij} is the probability of a dart (directed edge) from vertex i to vertex j, $i \neq j$ (contrary to the direction of darts in Tutte 1984), and all darts occur independently. If P is the (possibly asymmetric) matrix of dart probabilities, with 0 diagonal elements, then (Cohen 1986)

$$E(T_1|P) = \det\{[K(P)](1)\}. \tag{3}$$

Fix any i and j with $i \neq j$ and let $\delta_{gh}=1$ if $g=h$, $\delta_{gh}=0$ if $g \neq h$. Then elementary calculations similar to those above lead to

$$\partial E(T_1|P)/\partial p_{ij}$$
$$= (1-\delta_{1i})\{\det\{[K(P)](1,i)\} \tag{4}$$
$$+(\delta_{1j}-1)(-1)^{i+j}\det\{[K(P)](1,i;\ 1,j)\}\},$$

where $[K(P)]$ $(1, i; 1, j)$ denotes the $(n-2)\times(n-2)$ matrix that remains after deleting rows 1 and i and columns 1 and j from $K(P)$. This formula shows that $E(T_1|P)$ is completely insensitive to changes in the probabilities p_{1h} of darts from vertex 1 to any other vertex, as expected.

To illustrate, (4) gives for $n=3$

$$\partial E(T_1|P)/\partial p_{21} = p_{31}+p_{32},$$
$$\partial E(T_1|P)/\partial p_{23} = p_{31},$$

both of which follow by differentiating (3), namely,

$$E(T_1|P) = p_{21}p_{31}+p_{23}p_{31}+p_{21}p_{32}.$$

5. Total sensitivity of random graphs increases with evenness of probabilities

As in sections 1, 2 and 3, let P be a symmetric matrix of edge probabilities for anisotropic random graphs (with three or more vertices) and let $S(\alpha)$ be the sum of the sensitivities when the edge probability matrix is P_α.

If x and y are two real n-vectors, $x=(x_1, \dots, x_n)^T$, $y=(y_1, \dots, y_n)^T$, let $x_{[1]} \geqslant \dots \geqslant x_{[n]}$ denote the elements of x in decreasing order, and similarly for y. Following Marshall and Olkin (1979), say that x is majorized by y and write

$$x \prec y \text{ if and only if } \sum_{i=1}^{k} x_{[i]} < \sum_{i=1}^{k} y_{[i]},$$
$$k=1, \dots, n-1,$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}.$$

Theorem. *For $n=3$, $S(\alpha)$ is constant in α. For $n>3$, $S(\alpha)$ is strictly increasing in $\alpha \in [0, 1]$ to a maximum of*

$$S(1)=(n-1)[p(P)n]^{n-2}.$$

Proof. By (2), $S(\alpha)$ is the sum of all principal minors of order $n-2$ of $K(P_\alpha)$, which is identical to the $(n-2)$nd elementary symmetric function (ESF) of the eigenvalues of $K(P_\alpha)$ (e.g., Marshall and Olkin 1979, p. 504). The $(n-1)$-vector $\underset{\sim}{\mu}(\alpha)=(\mu_1(\alpha), \dots, \mu_{n-1}(\alpha))^T$ of the $n-1$ largest eigenvalues of $K(P_\alpha)$, under the labelling $\mu_1(\alpha) \geqslant \mu_2(\alpha) \geqslant \dots \geqslant \mu_{n-1}(\alpha) \geqslant \mu_n(\alpha)=0$, has strictly positive elements when $\alpha>0$ (Cohen 1986). If $n=3$, the first ESF is just the trace of $K(P_\alpha)$, which is a constant, equal to the trace of $K(P)$ by construction. If $n>3$, then the $(n-2)$nd ESF is a strictly Schur-concave function of $\underset{\sim}{\mu}(\alpha)$ (a fact attributed to I. Schur by Marshall and Olkin 1979, p. 78). All terms in the ESF that contain $\mu_n(\alpha)=0$ as a factor vanish. Combining Lemmas 3.1 and 3.3 of Cohen (1986) shows that, if μ_i is the ith biggest eigenvalue of $K(P)$, then $\mu_i(\alpha)=(1-\alpha)\mu_i+\alpha p(P)n$, for $i=1, \dots, n-1$, and if $0<\alpha_1<\alpha_2\leqslant 1$, then $\underset{\sim}{\mu}(\alpha_2) \prec \underset{\sim}{\mu}(\alpha_1)$ but $\underset{\sim}{\mu}(\alpha_2)$ is not a permutation of $\underset{\sim}{\mu}(\alpha_1)$. By strict Schur-concavity of the $(n-2)$nd ESF, $S(\alpha_1)<S(\alpha_2)$.

In the isotropic case, all $n-1$ elements of the vector $\underset{\sim}{\mu}$ (1) are equal to $p(P)n$, so the $(n-2)$nd ESF is $[p(P)n]^{n-2}\binom{n-1}{n-2}=(n-1)[p(P)n]^{n-2}$. □

Since, for every $i=1, \ldots, n$, $\det\{[K(P_\alpha)](i)\}$ increases strictly with $\alpha \in [0, 1]$ when P is anisotropic (Cohen 1986, Cor. 3.3), and since the Theorem just proved shows that $S(\alpha)$ also increases with α in this case, it is natural to ask whether, for all $1 \leq i < j \leq n$, $\det\{[K(P_\alpha)](i,j)\}$ also increases with $\alpha \in [0, 1]$. It is easy to construct numerical examples that show that, for some $i<j$, $\det\{[K(P_\alpha)](i,j)\}$ may decrease strictly as α increases.

6. Relative sensitivity of random graphs: a conjecture about majorization

Let $r_{ij}(\alpha)$ be the relative sensitivity of $E(T|P)$ to the probability of edge $\{i,j\}$, that is,

$$r_{ij}(\alpha) = (\partial E(T|P_\alpha)/\partial p_{ij})/S(\alpha).$$

Let $\underset{\sim}{r}(\alpha)$ be the $\binom{n}{2}$-vector of relative sensitivities $r_{ij}(\alpha)$ in some order, such as lexicographically by subscript.

Conjecture. *If the matrix P of edge probabilities is anisotropic, then*

$$0 \leq \alpha_1 < \alpha_2 \leq 1 \textit{ implies } \underset{\sim}{r}(\alpha_2) \prec \underset{\sim}{r}(\alpha_1)$$

and $\underset{\sim}{r}(\alpha_1)$ *is not a permutation of* $\underset{\sim}{r}(\alpha_2)$.

The conjecture asserts that the nearer P_α is to an isotropic matrix, the more nearly equal are the importance factors of the different edges.

Proof of conjecture for $n=3$. By direct computation from (2),

$$\partial E(T|P_\alpha)/\partial p_{12} = (1-\alpha)(p_{31}+p_{32}) + 2\alpha p(P),$$
$$\partial E(T|P_\alpha)/\partial p_{13} = (1-\alpha)(p_{21}+p_{23}) + 2\alpha p(P),$$
$$\partial E(T|P_\alpha)/\partial p_{23} = (1-\alpha)(p_{12}+p_{13}) + 2\alpha p(P).$$

The sum of these sensitivities, $S(\alpha)$, equals the sum of the elements of P or trace $K(P_\alpha)$, which is independent of α and is positive. Call the sum c. Thus $c\underset{\sim}{r}(\alpha)$ is the vector of sensitivities and has the form $(1-\alpha)x + \alpha\bar{x}$, where x is the non-zero non-negative vector $(p_{31}+p_{32}, p_{21}+p_{23}, p_{12}+p_{13})$ and $\bar{x}$ is the vector with all elements equal to the mean $2p(P)$ of the elements of x. It follows that $\underset{\sim}{r}(\alpha_2) \prec \underset{\sim}{r}(\alpha_1)$ by a very slight generalization (from positive vectors to non-negative vectors) of Lemma 3.1 of Cohen (1986). Since not all elements of x can be equal, $\underset{\sim}{r}(\alpha_2)$ cannot be a permutation of $\underset{\sim}{r}(\alpha_1)$. □

The conjecture for general n is buttressed by at least one hundred examples with randomly chosen P and $\alpha = 0, 1/4, 1/2, 3/4, 1$ for each of $n = 4, 5$ and 6.

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