Math. Proc. Camb. Phil. Soc. (1987), **101**, 209
Printed in Great Britain

Arithmetic–geometric means of positive matrices

BY JOEL E. COHEN
Rockefeller University, New York, NY 10021, U.S.A.

AND ROGER D. NUSSBAUM
Rutgers University, New Brunswick, NJ 08903, U.S.A.

(*Received 23 April* 1986)

Abstract

We prove the existence of unique limits and establish inequalities for matrix generalizations of the arithmetic–geometric mean of Lagrange and Gauss. For example, for a matrix $A = (a_{ij})$ with positive elements a_{ij}, define (contrary to custom) $A^{\frac{1}{2}}$ elementwise by $[A^{\frac{1}{2}}]_{ij} = (a_{ij})^{\frac{1}{2}}$. Let $A(0)$ and $B(0)$ be $d\times d$ matrices $(1 < d < \infty)$ with all elements positive real numbers. Let $A(n+1) = (A(n)+B(n))/2$ and $B(n+1) = (d^{-1}A(n)\,B(n))^{\frac{1}{2}}$. Then all elements of $A(n)$ and $B(n)$ approach a common positive limit L. When $A(0)$ and $B(0)$ are both row-stochastic or both column-stochastic, dL is less than or equal to the arithmetic average of the spectral radii of $A(0)$ and $B(0)$.

1. *A limit theorem for positive matrices*

Let a and b be positive numbers. Define the sequences $\mathbf{a} = \{a(n)\}_{n=0}^{\infty}$ and $\mathbf{b} = \{b(n)\}_{n=0}^{\infty}$ by

$$\left.\begin{aligned} &a(0) = a, \quad b(0) = b,\\ &a(n+1) = (a(n)+b(n))/2, \quad b(n+1) = (a(n)\,b(n))^{\frac{1}{2}} \quad (n = 0, 1, \ldots).\end{aligned}\right\} \tag{1}$$

Because
$$\max(a,b) \geqslant (a+b)/2 \geqslant (ab)^{\frac{1}{2}} \geqslant \min(a,b) > 0,$$

with strict inequality everywhere if $a \neq b$, the sequences $\mathbf{a}$ and $\mathbf{b}$ each have a limit and it is the same limit. Denote the common limit by $M(a,b)$.

According to Cox [**3**], Lagrange defined the sequences (1) in 1785, noted that they have a common limit $M(a,b)$, and showed how to use them to compute elliptic integrals. In 1791, Gauss, then 14, independently discovered the sequences (1) and defined $M(a,b)$ to be the arithmetic–geometric mean, which he abbreviated to agM, of two positive numbers a, b. The agM has deep connections with elliptic integrals and diverse applications [**1–3**].

The iteration (1) applies as well to complex numbers a and b with positive real parts; the square root can be chosen so that every $b(n)$ has a positive real part. Stickel [**7**] established the convergence of the iteration (1) when a is I, the $d\times d$ identity matrix, and b is any $d\times d$ complex matrix, the eigenvalues of which all have positive real parts. The matrix square root is chosen so that the eigenvalues of every $b(n)$ have positive real parts.

Here we propose some different matrix generalizations of the iteration (1), show that they converge, and establish some inequalities governing the limiting values. Additional exact results and inequalities concerning the iteration (3) below have been established by P. D. Borwein and E. U. Stickel (personal communication, 4 March 1986) on the basis of a previous draft of this manuscript and correspondence.

For a fixed positive integer d, let A and B be $d \times d$ positive matrices, i.e. matrices in which all elements are positive numbers. For such $A = (a_{ij})$, define $A^{\frac{1}{2}}$ by $[A^{\frac{1}{2}}]_{ij} = (a_{ij})^{\frac{1}{2}}$. This elementwise square root is not the usual definition of the square root of a non-singular matrix (e.g. [**5**]). Clearly $A^{\frac{1}{2}} A^{\frac{1}{2}} \neq A$, but $A^{\frac{1}{2}} * A^{\frac{1}{2}} = A$, where $*$ denotes the Schur or Hadamard product, i.e. elementwise multiplication ($[A * B]_{ij} = a_{ij} b_{ij}$).

If the sequences $\mathbf{A}^* = \{A(n)\}_{n=0}^{\infty}$, $\mathbf{B}^* = \{B(n)\}_{n=0}^{\infty}$ of $d \times d$ positive matrices are defined by

$$\left.\begin{aligned} &A(0) = A, \quad B(0) = B, \\ &A(n+1) = (A(n)+B(n))/2, \quad B(n+1) = (A(n) * B(n))^{\frac{1}{2}}, \end{aligned}\right\} \tag{2}$$

then obviously $\mathbf{A}^*$ and $\mathbf{B}^*$ have the common limit $M^*(A, B)$ with elements $[M^*(A,B)]_{ij} = M(a_{ij}, b_{ij})$.

The purpose of this note is to define and describe the limiting behaviour of slightly less trivial generalizations of the agM for positive matrices. Define the sequences $\{A(n)\}_{n=0}^{\infty}$, $\{B(n)\}_{n=0}^{\infty}$ of $d \times d$ positive matrices by

$$\left.\begin{aligned} &A(0) = A, \quad B(0) = B \\ &A(n+1) = (A(n)+B(n))/2, \quad B(n+1) = (d^{-1}A(n)\, B(n))^{\frac{1}{2}}, \end{aligned}\right\} \tag{3}$$

where AB denotes the ordinary matrix product A times B. (When $d = 1$, (2) and (3) both reduce to (1).) Let J be the $d \times d$ matrix in which every element is 1.

THEOREM 1. *There exists a positive number $\mu(A, B)$ such that*

$$\lim_{n\uparrow\infty} A(n) = \lim_{n\uparrow\infty} B(n) = \mu(A, B) J \equiv M(A, B), \tag{4}$$

and convergence to the limit is geometrically fast.

Before proving (4), we note that if $A(n) = B(n) = cJ$ for some positive number, c, then $A(n+1) = B(n+1) = cJ$, so cJ is a 'steady-state' solution of (3). The task is to prove that every solution of (3) converges to this steady-state solution with c depending on A and B.

It is also easy to see, as David A. Cox pointed out (personal communication, 23 January 1986) that $\mu(aJ, bJ) = M(a, b)$ when a and b are positive real numbers.

Proof of Theorem 1. Consider the directed graph D with $2d^2$ vertices labelled by C_{ij} where C takes the values $C = A$ or $C = B$, and $i = 1, \ldots, d$, and $j = 1, \ldots, d$. Each vertex of D represents one of the $2d^2$ elements of the matrices A and B. Let there be a directed edge to $(C_1)_{ij}$ from $(C_2)_{gh}$, and write $(C_1)_{ij} \leftarrow (C_2)_{gh}$, if and only if, according to (3),

$$\partial[C_1(n+1)]_{ij} / \partial[C_2(n)]_{gh} > 0.$$

For example, $A_{11} \leftarrow B_{11}$ because $A_{11}(n+1) = (A_{11}(n) + B_{11}(n))/2$ so B_{11} at step n influences A_{11} at step $n+1$. Similarly, for $i = 1, \ldots, d$, $B_{11} \leftarrow B_{i1}$.

It follows from the chain rule that D has a directed path of $k \geqslant 1$ directed edges from $(C_2)_{gh}$ to $(C_1)_{ij}$, and we write $(C_1)_{ij} \leftarrow_k (C_2)_{gh}$, if and only if

$$\partial[C_1(n+k)]_{ij}/\partial[C_2(n)]_{gh} > 0.$$

Returning to the preceding examples, $A_{11} \leftarrow_2 B_{i1}$, for $i = 1, \ldots, d$, i.e. A_{11} at step $n+2$ is influenced by or depends on every element of the first column of B at step n.

We next show that D is strongly connected, i.e., that there is a directed path from any vertex to any other. Since, for any $i = 1, \ldots, d$, we have $B_{i1} \leftarrow A_{ij}$ for all $j = 1, \ldots, d$; and since $A_{ij} \leftarrow B_{ij}$ for all $i, j = 1, \ldots, d$, we have the following paths:

$$\left.\begin{array}{c} A_{11} \leftarrow_3 A_{ij} \quad \text{and} \quad A_{11} \leftarrow_4 B_{ij}, \\ B_{11} \leftarrow_2 A_{ij} \quad \text{and} \quad B_{11} \leftarrow_3 B_{ij}, \\ (\text{for all } i, j = 1, \ldots, d). \end{array}\right\} \tag{5}$$

Since the choice of the $(1, 1)$ element of A or B is arbitrary, obviously the same is true for any other element of A or B. Thus D is strongly connected.

Moreover, $A_{ij} \leftarrow A_{ij}$, for all i, $j = 1, \ldots, d$. Therefore for all $k \geqslant 4$, for all g, h, i, $j = 1, \ldots, d$, and for C_1, $C_2 = A$ or B,

$$(C_1)_{ij} \leftarrow_k (C_2)_{gh}.$$

For $k \geqslant 4$, each element of $C_1(n+k)$ depends simultaneously on *all* elements of $C_2(n)$.

Define the upper and lower bounds, for $n = 0, 1, \ldots,$

$$U(n) = \max_{i,j} \max\,(A_{ij}(n), B_{ij}(n)),$$

$$L(n) = \min_{i,j} \min\,(A_{ij}(n), B_{ij}(n)).$$

Then, from (3),

$$U(n) \geqslant U(n+1) \geqslant L(n+1) \geqslant L(n), \tag{6}$$

so the limits $L = \lim_{n\uparrow\infty} L(n)$, $U = \lim_{n\uparrow\infty} U(n)$ exist and $U \geqslant L$.

We now show that $U = L$ and that $U(n)$ and $L(n)$ converge to their common limit geometrically fast.

Fix n and let $U^* = U(n)$, $L^* = L(n)$. Since each element of $A(n+1)$ and $B(n+1)$ is a monotone non-decreasing function of each element of $A(n)$ and $B(n)$, $U(n+1)$, the largest of the elements of $A(n+1)$ and $B(n+1)$, will be large as possible if some single element of $A(n)$ or $B(n)$ equals L^* and all the remaining elements of $A(n)$ and $B(n)$ equal U^*. So suppose this is true. If $A_{ij}(n+1)$ and $B_{ij}(n+1)$ are elements of $A(n+1)$ and $B(n+1)$ that depend on the element of $A(n)$ or $B(n)$ that is equal to L^*, then

$$A_{ij}(n+1) = (L^* + U^*)/2 = U^* - (U^* - L^*)/2,$$

$$B_{ij}(n+1) = [d^{-1}\{(d-1)\,U^{*2} + U^* L^*\}]^{\frac{1}{2}} = [U^*\{(1-1/d)\,U^* + L^*/d\}]^{\frac{1}{2}}$$

$$\leqslant U^*/2 + \{(1-1/d)\,U^* + L^*/d\}/2 = U^* - (U^* - L^*)/2d,$$

using the inequality of arithmetic and geometric means. Thus every element of $A(n+1)$ and $B(n+1)$ that depends on the L^* element of $A(n)$ or $B(n)$ is not greater than $U^* - (U^* - L^*)/(2d)$. (The other elements of $A(n+1)$ and $B(n+1)$ could be as large as U^*.) By iteration, every element of $A(n+2)$ and $B(n+2)$ that depends on any element of $A(n+1)$ or $B(n+1)$ that in turn depends on the L^* element of $A(n)$ or $B(n)$ is not greater than

$$U^* - (U^* - [U^* - (U^* - L^*)/(2d)])/(2d) = U^* - (U^* - L^*)/(2d)^2.$$

Iterating two more steps, we observe that *every* element of $A(n+4)$ and $B(n+4)$ depends on the L^* element of $A(n)$ or $B(n)$ (via a path given by the directed graph D) and is not greater than $U^* - (U^* - L^*)/(2d)^4$, that is,

$$U(n+4) \leqslant U(n) - (U(n) - L(n))/(2d)^4.$$

From the last inequality of (6), we then have

$$U(n+4) - L(n+4) < [U(n) - L(n)]\,(1 - (2d)^{-4}).$$

Hence $U = L$ and, in the notation of (4), both equal $\mu(A, B)$. ∎

Since generally $AB \neq BA$, generally $\mu(A,B) \neq \mu(B,A)$.

David A. Cox (personal communication, 23 January 1986) points out the following amusing corollary of Theorem 1. If A is a $d \times d$ positive matrix such that $(A^2)^{\frac{1}{2}} = d^{\frac{1}{2}}A$, then, for some $c > 0$, $A = cJ$. His proof is that if $A(0) = B(0) = A$, then $A(n) = B(n) = A$ for all n. A direct proof of a stronger result involving the map $A \to (d^{-1}A^2)^{\frac{1}{2}}$ is sketched in the next section.

2. *A general principle for the existence of a limit*

To keep this paper self-contained, we have given an *ad hoc* proof of the existence of $\mu(A,B)$. In this section, we explain how the results of Section 1 are a special case of a general theorem of Nussbaum [**6**]. That general theorem also contains other generalizations of the agM, for example, one of Everett and Metropolis [**4**]. We now describe the general theorem and its relation to our problem.

A closed, convex subset K of a Banach space X will be called a cone if $tx \in K$ for all $t \geqslant 0$ and $x \in K$ and if $x \in K - \{0\}$ implies that $-x \notin K$. A cone induces a partial ordering by $x \leqslant y$ if $y - x \in K$. If the interior, K^0, of K is non-empty and $f: K^0 \to K^0$ is a map, f is called *order-preserving* if $f(x) \leqslant f(y)$ whenever $x \leqslant y$; and f is *homogeneous of degree 1* if $f(tx) = tf(x)$ for all $t > 0$ and $x \in K$.

THEOREM 2. (See theorem 3·2 in [**6**].) *Let K be a cone with non-empty interior in a finite-dimensional Banach space X and let $f: K^0 \to K^0$ be a continuous, order-preserving map which is homogeneous of degree* 1. *Assume that there exists $v \in K^0$ such that $f(v) = v$ and that f is C^1 on an open neighbourhood of v. Let $L = f'(v)$ be the Fréchet derivative of f at v. Assume that there exists an integer $m \geqslant 1$ such that for each $x \in K - \{0\}$, $L^m x \in K^0$. Then for each $x \in G$, there exists a positive number $\mu(x)$ such that*

$$\lim_{n \to \infty} f^n(x) = \mu(x)\, v,$$

where f^n denotes the composition of f with itself n times. The map $x \to \mu(x)$ is continuous, C^1 on an open neighbourhood of v, order-preserving and homogeneous of degree 1. *If $v^* \in X^*$ denotes the Fréchet derivative $\mu'(v)$ of μ at v, then v^* is the unique element of X^* such that $L^* v^* = v^*$ and $v^*(v) = 1$, where L^* is the Banach space adjoint of L. If f is C^k (real analytic) on K^0, then μ is C^k (real analytic) on K^0.*

A version of Theorem 2 is also proved in [**6**] for general Banach spaces.

The results of [6] also imply that the convergence in Theorem 2 is geometric: given a compact subset M of K^0, there exist constants B and c, $0 < c < 1$, depending on M, such that $\|f^n(x) - \mu(x)v\| \leqslant Bc^n$, for all $x \in M$.

Theorem 2 immediately implies that f has a unique (to within positive scalar multiples) eigenvector in K^0. However, if f extends continuously to K it may well have other eigenvectors in the boundary of K. This will be the case for f given by (3).

To apply Theorem 2 to (3), let X denote the Banach space of ordered pairs (A, B) of $d \times d$ matrices. Let K denote the cone of ordered pairs (A, B) in X all of whose components are non-negative. Define $f: K^0 \to K^0$ by

$$f(A,B) = ((A+B)/2,\ (d^{-1}AB)^{\frac{1}{2}}), \tag{7}$$

where, as before, the square root in (7) is the elementwise square root. It is easy to check that f is real analytic on K^0, order-preserving, and homogeneous of degree 1 and that $f(J,J) = (J,J)$ (where J has every element 1). An easy application of the chain rule shows that the Fréchet derivative of f at (J,J) is the linear map $L: X \to X$ given by

$$L(A,B) = ((A+B)/2,\ (2d)^{-1}(AJ+JB)). \tag{8}$$

To see that L satisfies the hypotheses of Theorem 2, if $(A,B) \in K - \{0\}$, define $(A_k, B_k) = L^k(A,B)$. Since A and B are not both 0, A_1 has some positive entry, say in row i. Equation (8) then implies that all entries in row i of B_2 are positive. Thus, JB_2 and B_3 have all entries positive; and finally, A_4 and B_4 have all entries positive, i.e. $L^4(A,B) \in K^0$.

Theorem 2 now implies that

$$\lim_{n\to\infty} f^n(A,B) = \mu(A,B)\,(J,J) \quad (\mu(A,B) > 0),$$

for all $(A,B) \in K^0$, which is just Theorem 1. Furthermore, Theorem 2 implies that $(A,B) \to \mu(A,B)$ is real analytic. Let $\mathbf{1}$ be the d-vector with all elements 1. If

$$v^*(A,B) = (2d^2)^{-1} \sum_{i,j} (a_{ij} + b_{ij}) = (2d^2)^{-1}\mathbf{1}^T(A+B)\,\mathbf{1}, \tag{9}$$

then $v^* \in X^*$, $v^*(J,J) = 1$ and $L^*v^* = v^*$, so v^* is the Fréchet derivative of μ at (J,J).

Two other maps are closely related to f. First, define $g: K^0 \to K^0$ by

$$g(A,B) = ((A+B)/2,\quad ((2d)^{-1}(AB+BA))^{\frac{1}{2}}). \tag{10}$$

From Theorem 2 or the kind of argument employed in Section 1, one obtains

$$\lim_{n\to\infty} g^n(A,B) = \xi(A,B)\,(J,J) \quad (\xi(A,B) > 0), \tag{11}$$

for all $(A,B) \in K^0$. The map $(A,B) \to \xi(A,B)$ is real analytic, homogeneous of degree 1, order-preserving and has the same Fréchet derivative as f at (J,J). In addition, $\xi(A,B) = \xi(B,A)$.

Second, if Y denotes the Banach space of $d \times d$ matrices and C is the cone of non-negative matrices, define $h: C^0 \to C^0$ by

$$h(A) = (d^{-1}A^2)^{\frac{1}{2}}, \tag{12}$$

where again the square root denotes the elementwise square root. The map h satisfies all assumptions of Theorem 2 and $h(J) = J$, so for each $A \in C^0$ there exists $\lambda(A) > 0$ such that

$$\lim_{n\to\infty} h^n(A) = \lambda(A)\,J. \tag{13}$$

Equation (13) implies that h has a unique normalized eigenvector in C^0, which is basically the observation made at the end of Section 1.

3. *Estimates for* $\mu(A,B)$, $\xi(A,B)$ *and* $\lambda(A)$

We now give some estimates for $\mu(A,B)$, $\xi(A,B)$ and $\lambda(A)$. In certain important cases, our estimates use $r(A)$ and $r(B)$, the spectral radii of A and B, respectively.

First, we renormalize. If J is the $d\times d$ matrix with all elements 1, define J_1 by

$$J_1 = d^{-1}J. \tag{14}$$

Then $J_1\mathbf{1} = \mathbf{1}$, so the theory of positive matrices implies $r(J_1) = 1$. If A and B are positive $d\times d$ matrices and f, g and h are as defined in (7), (10) and (12), our previous results imply that there are positive numbers $\mu_1(A,B)$, $\xi_1(A,B)$ and $\lambda_1(A)$ such that

$$\left.\begin{aligned} \lim_{n\to\infty} f^n(A,B) &= \mu_1(A,B)\,(J_1, J_1),\\ \lim_{n\to\infty} g^n(A,B) &= \xi_1(A,B)\,(J_1, J_1),\\ \lim_{n\to\infty} h^n(A) &= \lambda_1(A)\,J_1. \end{aligned}\right\} \tag{15}$$

Obviously $\mu_1(A,B) = d\mu(A,B)$, etc.

LEMMA 1. *Let $A\in C$ and $B\in C$, i.e. A and B are non-negative $d\times d$ matrices. Let x be a non-negative $d\times 1$ column vector and $x^{\frac12}$ be its elementwise square root. Let $E = (d^{-1}AB)^{\frac12}$ and $F = ((2d)^{-1}(AB+BA))^{\frac12}$. If, for y and $z\in\mathbb{R}^d$, one writes $y \leqslant z$ when $y_i \leqslant z_i$ for $1\leqslant i\leqslant d$, and if one has*

$$Ax \leqslant \alpha x \quad \textit{and} \quad Bx \leqslant \beta x$$

for positive scalars α and β, then

$$E(x^{\frac12}) \leqslant (\alpha\beta)^{\frac12}x^{\frac12} \quad \textit{and} \quad F(x^{\frac12}) \leqslant (\alpha\beta)^{\frac12}x^{\frac12}.$$

Analogous conclusions are true if x is a $1\times d$ row vector and $xA \leqslant \alpha x$ and $xB\leqslant \beta x$.

Proof. We shall prove the theorem for the matrix E. The proof for F is essentially the same.

The ith component $(Ex^{\frac12})_i$ of $Ex^{\frac12}$ is given by

$$(Ex^{\frac12})_i = \sum_{k=1}^{d}\left(d^{-1}\sum_{j=1}^{d} a_{ij}\,b_{jk}\,x_k\right)^{\frac12}. \tag{16}$$

Applying the Cauchy–Schwartz inequality to the right side of (16) gives

$$\left.\begin{aligned} (Ex^{\frac12})_i &\leqslant d^{-\frac12}\left(\sum_{k=1}^{d} 1^2\right)^{\frac12}\left(\sum_{k=1}^{d}\sum_{j=1}^{d} a_{ij}\,b_{jk}\,x_k\right)^{\frac12}\\ &= \left(\sum_{j=1}^{d} a_{ij}\sum_{k=1}^{d} b_{jk}\,x_k\right)^{\frac12}\\ &\leqslant \left(\sum_{j=1}^{d} a_{ij}\,\beta x_j\right)^{\frac12} \leqslant (\alpha\beta)^{\frac12}x_i^{\frac12}. \quad \blacksquare \end{aligned}\right\} \tag{17}$$

A non-negative square matrix A is *primitive* if, for some $m > 1$, all entries of A^m are positive.

THEOREM 3. *Let C be the cone of non-negative $d \times d$ matrices (as before) and let $h: C \to C$ be defined as in* (12). *If $A \in C$ is a primitive matrix, then*

$$\lim_{n\to\infty} h^n(A) = \lambda_1(A)\,J_1, \tag{18}$$

where $\lambda_1(A)$ is positive real and J_1 is as in (14). *Furthermore,*

$$\lambda_1(A) \leqslant r(A), \tag{19}$$

where $r(A)$ is the spectral radius of A. Equality holds in (19) *if $A = J_1$.*

Proof. If $A \in C^0$, (18) was established in Section 2. However, if A^m has all positive entries and k is such that $2^k \geqslant m$, it is easy to show that $h^k(A) \in C^0$, so the existence of the limit in this case follows from the other case and $\lambda_1(A) = \lambda_1(h^k(A)) > 0$.

The theory of non-negative matrices implies that if B is primitive, B has an eigenvector with all components positive and with eigenvalue $r(B)$. Furthermore, if B is any non-negative, square matrix and $By \leqslant \beta y$ for some vector y with all components positive, then $r(B) \leqslant \beta$.

In our case, A has a strictly positive eigenvector with eigenvalue $\alpha = r(A)$. If $A_n = h^n(A)$, Lemma 1 implies (taking $A = B$)

$$A_1\, x^{\frac{1}{2}} \leqslant \alpha x^{\frac{1}{2}},$$

and using Lemma 1 repeatedly one obtains

$$A_n(x^{\epsilon_n}) \leqslant \alpha x^{\epsilon_n}, \quad \text{where} \quad \epsilon_n = 2^{-n}.$$

The previous observations imply that

$$r(A_n) \leqslant \alpha = r(A),$$

and taking the limit as $n \to \infty$ gives

$$r(\lambda_1(A)\,J_1) = \lambda_1(A)\,r(J_1) = \lambda_1(A) \leqslant r(A). \quad \blacksquare$$

THEOREM 4. *Suppose that $A = (a_{ij})$ and $B = (b_{ij})$ are $d \times d$ positive matrices and define*

$$\left.\begin{aligned} \alpha &= \max_{1\leqslant i\leqslant d} \sum_{j=1}^{d} a_{ij}, \quad \beta = \max_{1\leqslant i\leqslant d} \sum_{j=1}^{d} b_{ij}, \\ \gamma &= \max_{1\leqslant j\leqslant d} \sum_{i=1}^{d} a_{ij} \quad \textit{and} \quad \delta = \max_{1\leqslant j\leqslant d} \sum_{i=1}^{d} b_{ij}. \end{aligned}\right\} \tag{20}$$

If $M(\alpha, \beta)$ is the arithmetic–geometric mean of α and β, then

$$\left.\begin{aligned} \mu_1(A,B) &\leqslant \min\,(M(\alpha,\beta),\, M(\gamma,\delta)), \\ \xi_1(A,B) &\leqslant \min\,(M(\alpha,\beta),\, M(\gamma,\delta)). \end{aligned}\right\} \tag{21}$$

Equality holds in (21) *if $A = \alpha J_1$ and $B = \beta J_1$.*

Proof. We shall prove inequality (21) for $\mu_1(A,B)$, since the argument for $\xi_1(A,B)$ is essentially the same. As before, let $\mathbf{1}$ denote the $d \times 1$ column vector with all entries 1 and $\mathbf{1}^T$ its transpose. Then $A\mathbf{1} \leqslant \alpha\mathbf{1}$ and $B\mathbf{1} \leqslant \beta\mathbf{1}$. Lemma 1 implies $A_1\mathbf{1} \leqslant ((\alpha+\beta)/2)\,\mathbf{1}$ and $B_1\mathbf{1} \leqslant (\alpha\beta)^{\frac{1}{2}}\mathbf{1}$, where $(A_1, B_1) = f(A,B)$ and f is as in (7).

Generally, if $\phi(\alpha,\beta) = ((\alpha+\beta)/2,\ (\alpha\beta)^{\frac{1}{2}})$ and $(\alpha_k,\beta_k) = \phi^k(a,b)$ and $(A_k, B_k) = f^k(A,B)$, then, by repeated applications of Lemma 1, we obtain $A_k\mathbf{1} \leqslant \alpha_k\mathbf{1}$ and $B_k\mathbf{1} \leqslant \beta_k\mathbf{1}$. Taking the limit as $k\to\infty$,

$$\mu_1(A,B)\,J_1\mathbf{1} = \mu_1(A,B)\,\mathbf{1} \leqslant M(\alpha,\beta)\,\mathbf{1},$$

so $\mu_1(A,B) \leqslant M(\alpha,\beta)$. Since $\mathbf{1}^T A \leqslant \gamma\mathbf{1}^T$ and $\mathbf{1}^T B \leqslant \delta\mathbf{1}^T$, the argument that $\mu_1(A,B) \leqslant M(\gamma,\delta)$ is completely analogous. ∎

If A is a $d\times d$ matrix, the formula

$$\|A\|_\infty = \max_{1\leqslant i\leqslant d}\sum_{j=1}^{d}|a_{ij}|$$

defines a norm. In fact, if one defines for a $d\times 1$ vector x the standard sup norm by

$$\|x\|_\infty = \max_{1\leqslant i\leqslant d}|x_i|,$$

then

$$\|A\|_\infty = \max\{\|Ax\|_\infty : \|x\|_\infty \leqslant 1\}.$$

Thus Theorem 4 provides an estimate for $\mu_1(A,B)$ in terms of the agM of $\|A\|_\infty$ and $\|B\|_\infty$.

There is another natural norm on the set of $d\times d$ matrices for which one obtains similar estimates. If A is a $d\times d$ matrix, define $\|A\|_{HS}$, the Hilbert-Schmidt norm of A, by

$$\|A\|_{HS} = \left(\sum_{i=1}^{d}\sum_{j=1}^{d}|a_{ij}|^2\right)^{\frac{1}{2}}.$$

It is well-known that $\|A\|_{HS}$ actually defines a norm.

LEMMA 2. *Let A and B be non-negative matrices, neither of which is identically zero. If $(A_1, B_1) = f(A,B)$ is defined by* (7), *then*

$$\|A_1\|_{HS} \leqslant (\tfrac{1}{2})\,(\|A\|_{HS} + \|B\|_{HS}), \tag{22}$$

$$\|B_1\|_{HS} \leqslant (\|A\|_{HS}\|B\|_{HS})^{\frac{1}{2}}. \tag{23}$$

Equality holds in (22) *if and only if $B = \alpha A$ for some $\alpha > 0$, and equality holds in* (23) *if and only if there exists $\beta > 0$ such that $a_{ij} = \beta b_{jk}$ for all i, j, k.*

Proof. For notational convenience, write $\|\,.\,\|$ for $\|\,.\,\|_{HS}$. Inequality (22) and the condition for equality follow immediately from the Cauchy–Schwartz inequality.

By definition

$$\|B_1\| = \left(\sum_{i,\,k}\left(d^{-1}\sum_{j=1}^{d}a_{ij}b_{jk}\right)\right)^{\frac{1}{2}}.$$

The Cauchy–Schwartz inequality gives

$$\sum_{j=1}^{d}a_{ij}b_{jk} \leqslant \left(\sum_{j=1}^{d}a_{ij}^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{d}b_{jk}^2\right)^{\frac{1}{2}}, \tag{24}$$

and equality holds in (24) if and only if $a_{ij} = 0$ for $1\leqslant j\leqslant d$ or there exists $\lambda_{ik} \geqslant 0$ such that

$$a_{ij} = \lambda_{ik}b_{jk} \quad\text{for}\quad 1\leqslant j\leqslant d. \tag{25}$$

If we define α_i and β_k by

$$\alpha_i = \left(\sum_{j=1}^{d} a_{ij}^2\right)^{\frac{1}{2}}, \quad \beta_k = \left(\sum_{j=1}^{d} b_{jk}^2\right)^{\frac{1}{2}},$$

inequality (24) gives

$$\|B_1\| \leqslant \left(\sum_{i,k} d^{-1}\alpha_i \beta_k\right)^{\frac{1}{2}} = \left(d^{-1}\left(\sum_{i=1}^{d} \alpha_i\right)\left(\sum_{k=1}^{d} \beta_k\right)\right)^{\frac{1}{2}}. \tag{26}$$

The Cauchy–Schwartz inequality implies that

$$\sum_{i=1}^{d} \alpha_i \leqslant d^{\frac{1}{2}}\left(\sum_{i=1}^{d} \alpha_i^2\right)^{\frac{1}{2}} = d^{\frac{1}{2}}\|A\|, \tag{27}$$

and equality holds in (27) if and only if all the α_i are equal (so none of the α_i equals zero). Similarly,

$$\sum_{k=1}^{k} \beta_k \leqslant d^{\frac{1}{2}}\left(\sum_{k=1}^{d} \beta_k^2\right)^{\frac{1}{2}} = d^{\frac{1}{2}}\|B\|, \tag{28}$$

and equality holds in (28) if and only if all the β_k are equal (and hence all non-zero). By substituting inequalities (27) and (28) in (26), we obtain inequality (23). Furthermore, our remarks show that equality holds in (23) if and only if all the α_i are equal and non-zero, all the β_k are equal and non-zero, and (25) holds. Using this information, one can easily see that λ_{ik} is a positive constant independent of i and k. ∎

COROLLARY 1. *Let C be the cone of non-negative $d \times d$ matrices, let $K = C \times C$ and let $f: K \to K$ be defined as in* (7). *If $(A_0, B_0) \in \partial K$ and $(A_k, B_k) = f^k(A_0, B_0) \in \partial K$ for all $k \geqslant 1$, then $\lim_{k\to\infty}(A_k, B_k) = (0,0)$.*

Proof. If β is such that $A \leqslant \beta J_1$ and $B \leqslant \beta J_1$, it is easy to see that $A_k \leqslant \beta J_1$ and $B_k \leqslant \beta J_1$ for all k. Lemma 2 implies that $\|A_k\| + \|B_k\|$ (where the norm is the Hilbert–Schmidt norm) is a decreasing function of k, so

$$\lim_{k\to\infty} \|A_k\| + \|B_k\| = L. \tag{29}$$

If the corollary is false, then (because A_k and B_k are bounded) we can select a subsequence (A_{k_i}, B_{k_i}) which converges to $(E, F) \in \partial K$, $(E,F) \neq (0,0)$. Equation (29) implies that if $(E_k, F_k) = f^k(E,F)$, then

$$\|E\| + \|F\| = L = \|E_k\| + \|F_k\|. \tag{30}$$

If E or F is 0, one easily sees that (E_k, F_k) converges to $(0,0)$, which contradicts (30) and the assumption that $L \neq 0$. If $E \neq 0$ and $F \neq 0$, Lemma 2 implies that equality can occur in (30) if and only if there exist positive numbers α and β such that $F_{ij} = \alpha E_{ij}$ and $\beta F_{jk} = E_{ij}$ for all i, j, k. These equations easily imply that there exist constants c and d such that $E_{ij} = c$ and $F_{ij} = d$ for all i, j. Since $(E, F) \in \partial K$, this is impossible unless $c = 0$ or $d = 0$, and we assumed before that $E \neq 0$ and $F \neq 0$. Thus we have obtained a contradiction, and the corollary is true. ∎

If $(A_0, B_0) \in K^0$, an examination of the argument in Corollary 1 shows that the same sort of argument proves that (A_k, B_k) approaches a positive multiple of (J, J), thus yielding a third approach to our basic Theorem 1.

By using Corollary 1 we can define $\mu(A,B)$ for all $(A,B)\in K$. If $(A_k,B_k)=f^k(A,B)\in\partial K$ for all $k\geqslant 1$, define $\mu(A,B)=0$; if there exists $k\geqslant 0$ such that $f^k(A,B)\in K^0$, define $\mu(A,B)=\mu(f^k(A,B))$. We already know that μ is real analytic on K^0 (see Theorem 2), and it is an easy exercise to show by using Corollary 1 that μ is continuous on K.

THEOREM 5. *Suppose that $A=(a_{ij})$ and $B=(b_{ij})$ are $d\times d$ positive matrices. If $M(\alpha,\beta)$ denotes the agM of positive numbers α and β, then*

$$\mu_1(A,B)\leqslant M(\|A\|_{HS},\|B\|_{HS}),$$

and

$$\xi_1(A,B)\leqslant M(\|A\|_{HS},\|B\|_{HS}).$$

Proof. We shall only prove the inequality for $\mu_1(A,B)$, since the argument for $\xi_1(A,B)$ is essentially the same. Again we write $\|\,.\,\|$ for $\|\,.\,\|_{HS}$.

If $(A_k,B_k)=f^k(A,B)$ and if ϕ is defined as in the proof of Theorem 4, Lemma 2 implies (in the obvious notation)

$$(\|A_1\|,\|B_1\|)\leqslant\phi(\|A\|,\|B\|).$$

Repeatedly using this inequality and Lemma 2, we obtain

$$(\|A_k\|,\|B_k\|)\leqslant\phi^k(\|A\|,\|B\|). \tag{31}$$

Since $\|J_1\|=1$, the left side of (31) approaches (μ_1,μ_1), where $\mu_1=\mu_1(A,B)$, and the right side approaches (M,M), where $M=M(\|A\|,\|B\|)$. This proves the theorem. ∎

If A and B are both multiples of row-stochastic matrices or both multiples of column-stochastic matrices, Theorem 4 implies that

$$\mu_1(A,B)\leqslant\tfrac{1}{2}(r(A)+r(B))\quad\text{and}\quad\mu_1(A,B)\leqslant r((A+B)/2). \tag{32}$$

If x is a vector, let $\|x\|_2$ denote the standard Euclidean norm $(\Sigma_i|x_i|^2)^{\frac{1}{2}}$ of x; and if C is a $d\times d$ matrix, define $\|C\|_2$ by

$$\|C\|_2=\sup\{\|Cx\|_2:\|x\|_2=1\}.$$

It is easy to show that

$$\|C\|_{HS}\leqslant d^{\frac{1}{2}}\|C\|_2\quad\text{and}\quad\|C\|_2\leqslant\|C\|_{HS}.$$

If C is symmetric, it is well-known that $\|C\|_2=r(C)$, so if A and B are symmetric matrices, all of whose entries are positive, Theorem 5 implies that

$$\mu_1(A,B)\leqslant M(\|A\|_{HS},\|B\|_{HS})\leqslant d^{\frac{1}{2}}M(r(A),r(B))\leqslant d^{\frac{1}{2}}\tfrac{1}{2}(r(A)+r(B)), \tag{33}$$

$$\mu_1(A,B)\leqslant d^{\frac{1}{2}}r((A+B)/2). \tag{34}$$

In view of inequalities (32) to (34), one might hope that there exists a constant c such that for all positive matrices A,

$$\mu(A,A)/r(A)\leqslant c. \tag{35}$$

Unfortunately, (35) is not true in general. To see this, define

$$A=\begin{pmatrix}1 & b\\ 1 & 1\end{pmatrix}$$

for $b \geqslant 1$ and define $(A_k, B_k) = f^k(A, A)$. If, at each stage one retains only the highest power of b in each entry, an elementary but tedious calculation shows that there exists $\delta > 0$ (δ independent of b for $b \geqslant 1$) such that

$$A_6 \geqslant \delta \begin{pmatrix} b^{\epsilon_1} & b^{\epsilon_2} \\ b^{\epsilon_3} & b^{\epsilon_4} \end{pmatrix} \quad \text{and} \quad B_6 \geqslant \delta \begin{pmatrix} b^{\epsilon_5} & b^{\epsilon_6} \\ b^{\epsilon_7} & b^{\epsilon_8} \end{pmatrix},$$

where $$\begin{pmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_3 & \epsilon_4 \end{pmatrix} = \begin{pmatrix} 23/32 & 1 \\ 17/32 & 17/32 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \epsilon_5 & \epsilon_6 \\ \epsilon_7 & \epsilon_8 \end{pmatrix} = \begin{pmatrix} 49/64 & 49/64 \\ 37/64 & 37/64 \end{pmatrix}.$$

It follows that for $b \geqslant 1$,

$$(A_6, B_6) \geqslant \delta b^{\epsilon}(J, J), \text{ where } \epsilon = 17/32.$$

Since $\mu(A_6, B_6) = \mu(A, A)$ and μ is order-preserving,

$$\mu(A, A) \geqslant \delta b^{\epsilon}.$$

On the other hand, we can directly solve for $r(A)$:

$$r(A) = 1 + b^{\frac{1}{2}},$$

so $$\lim_{b \to \infty} \mu(A, A)/r(A) = \infty.$$

Theorems 4 and 5 provide some estimates for $\mu_1(A, B)$ and $\xi_1(A, B)$ in terms of the agM of certain numbers and hence in terms of elliptic integrals. Stickel [7] established a connection between his matrix generalization of the arithmetic–geometric mean and elliptic integrals, and derived from this connection algorithms for computing the matrix exponential and matrix logarithm. It remains to be seen whether there are connections between $\mu(A, B)$, $\xi(A, B)$ and elliptic integrals that are deeper than the loose connections implied by Theorems 4 and 5.

J. E. C. was partially supported by NSF grant BSR 84-07461. R. D. N. was partially supported by NSF grant DMS 85-03316. We thank David A. Cox and Peter B. Borwein for helpful correspondence. J. E. C. is grateful for a fellowship from the John D. and Catherine T. MacArthur Foundation and the hospitality of Mr and Mrs William T. Golden during this work.

REFERENCES

[1] J. M. Borwein and P. B. Borwein. The arithmetic–geometric mean and fast computation of elementary functions. *SIAM Review* **26** (1984), 351–366.

[2] D. A. Cox. The arithmetic–geometric mean of Gauss. *Enseign. Math.* **30** (1984), 270–330.

[3] D. A. Cox. Gauss and the arithmetic–geometric mean. *Notices Amer. Math. Soc.* **32** (1985), 147–151.

[4] C. J. Everett and N. Metropolis. A generalization of the Gauss limit for iterated means. *Adv. in Math.* **7** (1971), 297–300.

[5] F. R. Gantmacher. *Theory of Matrices* (Chelsea, 1960).

[6] R. D. Nussbaum. Iterated nonlinear maps and Hilbert's projective metric, in preparation.

[7] E. U. Stickel. Fast computation of matrix exponential and logarithm. *Analysis (Muenchen)* **5** (1985), 163–173.