nality of the set of sets. Let S be the disjoint union of the above sets (any repeated u_i is regarded as a separate object); then cardinality $S=c$. But $\{u_i\}\subseteq S$, so cardinality $I\leqq$ cardinality S. Thus cardinality $I\leqq$ cardinality J and, by symmetry, cardinality $J\leqq$ cardinality I. Hence cardinality $I=$ cardinality J.

References

1. C. J. Everett, Jr., Vector spaces over rings, Bull. Amer. Math. Soc., 48 (1942) 312–316.
2. T. Fujiwara, Note on the isomorphism problem for free algebraic systems, Proc. Japan Acad., 31 (1955) 135–6.
3. N. Jacobson, Lectures in abstract algebra, vol. 2, Van Nostrand, New York, 1951.
4. W. G. Leavitt, Modules over rings of words, Proc. Amer. Math. Soc., 7 (1956) 188–193.

TWISTED DETERMINANTS THAT SUM TO ZERO

JOEL E. COHEN, Harvard University

Introduction. If $A=|a_{ji}|$ is a square matrix over the reals ($j=1,\cdots,n$, $i=1,\cdots,n$), and $V_j=(a_{j1},\cdots,a_{jn})$ is the jth row vector in A, let $s_j^1(V_j)=(a_{jn},a_{j1},\cdots,a_{j,n-1})$, $s_j^2(V_j)=(a_{j,n-1},a_{jn},a_{j1},\cdots,a_{j,n-2})$, etc. In general, for $1\leqq r\leqq n$, $s_j^r(a_{j1},\cdots,a_{jn})=(X_{j1}^{(r)},\cdots,X_{jn}^{(r)})$. To express $X_{ji}^{(r)}$ as a function of the original components of V_j, we introduce the following function e of any two natural numbers a and b; it is adapted from the ϵ in [2].

$$(1)\qquad \begin{aligned} e(a:b) &= 1 \quad \text{if } a \geqq b,\\ &= 0 \quad \text{if } a < b. \end{aligned}$$

Then

$$(2)\qquad X_{ji}^{(r)} = a_{j,i-r+n\cdot e(r:i)}.$$

Setting

$$(3)\qquad I(r) = i - r + n\cdot e(r:i),$$

we have $X_{ji}^{(r)}=a_{j,I(r)}$. By $s_j^r(A)$ we mean the matrix formed from A by applying s_j^r to the jth row vector and leaving all other rows as in A. Similarly, $s_j^r\times s_{j'}^{r'}(A)$ ($j\neq j'$) means the matrix formed by applying s_j^r to V_j, $s_{j'}^{r'}$ to $V_{j'}$, and leaving all other rows the same; and so on. We may thus specify exactly how a matrix is twisted. A twisted determinant is the determinant of the twisted matrix. The purpose of this note is to state a theorem and some corollaries about related twisted determinants that add to zero. These results generalize those stated in [1] and [3].

We recall that if a matrix has just one element a, then the determinant of the matrix is a. The determinant of the empty matrix (having no columns or rows) is 1 (see [2]).

THEOREM. *Let $A=|a_{ji}|$ be an n by n matrix ($n\geqq 2$), and $D(A)$ the determinant of A. Then*

$$(4)\qquad \sum_{r=1}^{n} D(s_1^r\times s_2^{n+1-r}(A)) = 0.$$

Proof. Let $A(1, 2|c_1, c_2)$ be the matrix formed from A by removing rows 1 and 2 and columns c_1 and c_2. Then by Laplace's expansion,

$$D(A) = \sum_{\substack{c_1=1 \\ c_2\neq c_1}}^{n} \pm \begin{vmatrix} a_{1,c_1} & a_{1,c_2} \\ a_{2,c_1} & a_{2,c_2} \end{vmatrix} D(A(1, 2 \mid c_1, c_2)).$$

The left-hand side of (4) equals

$$\begin{aligned} &\sum_{r=1}^{n} \sum_{\substack{c_1=1 \\ c_2\neq c_1}}^{n} \pm \begin{vmatrix} X_{1,c_1}^{(r)} & X_{1,c_2}^{(r)} \\ X_{2,c_1}^{(p)} & X_{2,c_2}^{(p)} \end{vmatrix} D(A(1, 2 \mid c_1, c_2)) \\ (5)\qquad &= \sum_{\substack{c_1=1 \\ c_2\neq c_1}}^{n} \pm \left(\sum_{r=1}^{n} \begin{vmatrix} X_{1,c_1}^{(r)} & X_{1,c_2}^{(r)} \\ X_{2,c_1}^{(p)} & X_{2,c_2}^{(p)} \end{vmatrix} \right) D(A(1, 2|c_1, c_2)), \end{aligned}$$

where

$$(6)\qquad p = n + 1 - r.$$

If we can show that the coefficient in (5) of $D(A(1, 2|c_1, c_2))$ is zero, for every (c_1, c_2), $c_1 \neq c_2$, then we have proved (4). Let c_1, c_2 be fixed at $c_1 = i$, $c_2 = j$, $1 \leqq i, j \leqq n$, $i \neq j$; i and j are fixed for the remainder of the proof. In analogy to (3), let

$$(7)\qquad J(r) = j - r + n \cdot e(r:j).$$

Obviously $I(p) = i - p + n \cdot e(p:i)$ and $J(p) = j - p + n \cdot e(p:j)$. Then by (2)

$$\begin{aligned} \sum_{r=1}^{n} \begin{vmatrix} X_{1,i}^{(r)} & X_{1,j}^{(r)} \\ X_{2,i}^{(p)} & X_{2,j}^{(p)} \end{vmatrix} &= \sum_{r=1}^{n} \begin{vmatrix} a_{1,I(r)} & a_{1,J(r)} \\ a_{2,I(p)} & a_{2,J(p)} \end{vmatrix} \\ (8)\qquad &= \sum_{r=1}^{n} a_{1,I(r)} a_{2,J(p)} - \sum_{r'=1}^{n} a_{1,J(r')} a_{2,I(p')}, \end{aligned}$$

a sum of $2n$ terms, where as before $p' = n + 1 - r'$. From (3) it is clear that as $r = 1, \cdots, i-1$, $I(r) = i-1, \cdots, 1$, and as $r = i, \cdots, n$, $I(r) = n, \cdots, i$. Similarly, as r runs from 1 to n, $J(r)$ ranges over 1 to n. Hence for each r there exists exactly one r' such that $I(r) = J(r')$, i.e., such that

$$(9)\qquad i - r + n \cdot e(r:i) = j - r' + n \cdot e(r':j).$$

If (9) implies that $I(p') = J(p)$, then we may rearrange the terms of (8) into pairs whose members differ only in sign. Then (8) becomes 0 and (4) is proved.

We now show that (9) implies $I(p') = J(p)$. From (9) we have

$$(10)\qquad r' = r + j - i + n(e(r':j) - e(r:i)).$$

There are three possibilities: $e(r':j) - e(r:i)$ can equal 0, $+1$, or -1.

Case I. If $e(r':j) - e(r:i) = 0$, then $r' = r + j - i$ and (6) gives $p' = p - j + i$. Then $I(p') = i - p' + n \cdot e(p':i) = i - p + j - i + n \cdot e(p - j + i:i) = j - p + n \cdot e(p:j)$

$=J(p)$. Here we use the property of e, that if a, b, c are integers, $e(a:b)=e(a+c:b+c)$.

Case II. If $e(r':j)-e(r:i)=1$, then $r'=r+j-i+n$. Using (6), $p'=p-j+i-n$. By exactly the same substitution as in Case I, $I(p')=j-p+n$. If we can show that $e(p:j)=1$, then $I(p')=j-p+n\cdot e(p:j)=J(p)$. Combining $p'=p-j+i-n$ and $p'=n+1-r'$, we get $p=(n-i)+(n-r')+1+j$, or $p>j$. Therefore $I(p')=J(p)$.

Case III. If $e(r':j)-e(r:i)=-1$, then $r'=r+j-i-n$ and $p'=p-j+i+n$. Then $I(p')=j-p=J(p)$ if $e(p:j)=0$. Combining $p'=p-j+i+n$ and $p'=n+1-r'$, we get $p=j+1-(i+r')$. But since $i\geqq 1$, and $r'\geqq 1$, $p<j$. Therefore $I(p')=J(p)$.

Hence all the terms of (8) cancel and the theorem is proved.

COROLLARY I.

$$\sum_{r=1}^{n} D(s_1^{r+k} \times s_2^{p+k}(A)) = 0, \qquad k = 1, \cdots n-1. \tag{11}$$

Adding k to r and to p simply rotates the top two rows together, leaving the rest of the matrix stationary. Since $s_j^{n+a}=s_j^a$ for integral a, (11) provides an alternative $n-1$ ways of expressing (4).

COROLLARY II.

$$\sum_{r=1}^{n} D(s_u^{r} \times s_v^{p}(A)) = 0, \qquad 1 \leqq u, v \leqq n, u \neq v. \tag{12}$$

This is immediate, since interchanging rows just changes the sign of the determinant.

COROLLARY III. *The obvious analogs of the theorem and first two corollaries hold for twists of column vectors instead of row vectors.*

References

1. J. E. Cohen, Further properties of third order determinants, this MAGAZINE, 35 (1962) 304.
2. R. M. Thrall and L. Tornheim, Vector spaces and matrices, Wiley, New York, 1957.
3. C. W. Trigg, A property of third order determinants, this MAGAZINE, 35 (1962) 78.

THE WAY OF REDEMPTION

M. D. TEPPER, Urbana, Illinois

Señor Bernardo Ruiz of Cádiz had found in Mr. Arnold an exceptionally shrewd man, a banker by profession, and a gentleman devoted as himself to the mystery of the cards. He decided it would add much to an already pleasant evening to share with him his latest amusement. Consequently, the sentimental señor took only the cards with hearts from the deck and arranged them in the fashion he desired, whereupon he took the first card and put it on the bottom of