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GROUPS AS MACHINES

Dear Sirs:

Yesterday I accidentally came across the article on the Fibonacci series in the November 1960 issue of *The Mathematics Student Journal*, page 3.

The independent work I have been doing is so closely allied that I thought you might be interested in it. I am a senior.

Sincerely,
JOEL E. COHEN

P.S. I would appreciate any suggestions.

Consider a machine with three possible states, such that the present state is determined by the two previous states. Let the table

		$t=-2$		
		I	A	B
	I	I	A	B
$t=-1$	A	A	B	I
	B	B	I	A

define the machine: I, A, and B, are the three states. To find the present state (at $t=0$) of the machine, find the column of the state at $t=-2$ and the row of the state at $t=-1$; their intersection is the present state.

From the table, the machine's pattern of behavior may be determined, starting arbitrarily with the states I and A. The pattern is a cycle of eight steps

$$I \quad A \quad A \quad B \quad I \quad B \quad B \quad A$$

which is then repeated. Starting with any pair of states (except I and I), the machine goes through this same cycle, since the cycle includes all possible pairs of states except I and I. When two adjacent states of the cycle are I and I, the machine gets "stuck" and repeats I.

The table defining the machine actually represents a finite group of order 3; since 3 is prime, the group is cyclic and the states I, A, and B may be represented in terms of powers of A. Hence $I=A^0$, $A=A^1$, and $B=A^2$. Substituting into the table we may drop the A's entirely and use only the exponents, getting

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

which is the group of integers modulo 3. The first rule given for finding the present state of the machine reduces to adding the exponents of A in the previous two states modulo 3. The cycle of behavior then becomes

$$0\;1\;1\;2\;0\;2\;2\;1.$$

If, however, the exponents are added, not modulo 3, the result is the Fibonacci series: 0 1 1 2 3 5 8 13 $\cdots$ with a zero prefixed. Hence the Fibonacci series mod 3 yields a cycle of 8 states. (This provides a partial answer to the question about the Fibonacci series mod p raised in *The Mathematics Student Journal*, 8(1): 3, November 1960.)

This characterization of a Fibonacci series mod p or of a cyclic group of prime order as a machine becomes interesting if considered as a special case of a Markoff chain process in which the mth symbol depends on the preceding $m-1$ (in this case $m=3$) and in which all the probabilities are either 0 or 1. As far as I know, groups have not yet been considered as defining determinate machines in automata theory, and their behavior patterns have not been investigated. I am currently comparing the behavior patterns or cycles of the five different groups of order 8.

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