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Formal Relationships 7

Life expectancy is the death-weighted average of the reciprocal of the survival-specific force of mortality

Joel E. Cohen

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Life expectancy is the death-weighted average of the reciprocal of the survival-specific force of mortality

Joel E. Cohen [1]

Abstract

The hazard of mortality is usually presented as a function of age, but can be defined as a function of the fraction of survivors. This definition enables us to derive new relationships for life expectancy. Specifically, in a life-table population with a positive age-specific force of mortality at all ages, the expectation of life at age x is the average of the reciprocal of the survival-specific force of mortality at ages after x, weighted by life-table deaths at each age after x, as shown in (6). Equivalently, the expectation of life when the surviving fraction in the life table is s is the average of the reciprocal of the survival-specific force of mortality over surviving proportions less than s, weighted by life-table deaths at surviving proportions less than s, as shown in (8). Application of these concepts to the 2004 life tables of the United States population and eight subpopulations shows that usually the younger the age at which survival falls to half (the median life length), the longer the life expectancy at that age, contrary to what would be expected from a negative exponential life table.

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1. Background and relationships

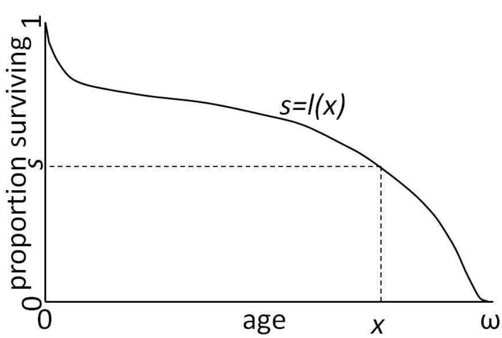
1.1 Background

The life table $\ell(x)$, constant in time, with continuous age x, is the proportion of a cohort (whether a birth cohort or a synthetic period cohort) that survives to age x or longer. In probabilistic terms, $\ell(x)$ is one minus the cumulative distribution function of length of life x. The maximum possible age ω may be finite or infinite. If $\omega = \infty$, then some individuals may live longer than any finite bound. By definition, $\ell(0) = 1$ and $\ell(\omega) = 0$. Assume $\ell(x)$ is a continuous, differentiable function of x, $0 \leq x \leq \omega$, and assume life expectancy at age 0 is finite. The age-specific force of mortality at age x is, by definition,

(1)
$$\mu(x) = -\frac{1}{\ell(x)}\frac{d\ell(x)}{dx}.$$

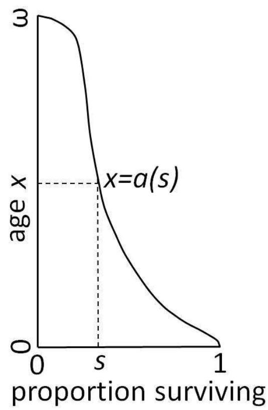
Assume $\mu(x) > 0$ for all $0 \leq x \leq \omega$. The life table $\ell(x)$ is strictly decreasing from $\ell(0) = 1$ to $\ell(\omega) = 0$ so there is a one-to-one correspondence between age x in $[0, \omega]$ and the proportion s in $[0, 1]$ of the cohort that survives to age x or longer. One direction of this correspondence is given by the life table function $s = \ell(x)$ (illustrated schematically in Figure 1 and for the United States population in 2004 in Figure 3A).

Figure 1: When the force of mortality is positive at every age x, the proportion surviving s, given by the life table according to $\mathrm{s} = \ell(\mathrm{x})$, strictly decreases as age x increases, so there is a one-to-one correspondence between age x and the proportion surviving s.



There appears to be no standard demographic term for the inverse function that maps the proportion surviving s, $0 \leq s \leq 1$, to the corresponding age x, so I propose to call it the age function a (illustrated schematically in Figure 2 and for the United States population in 2004 in Figure 3D). In words, the age $a(s)$ at which the fraction s of the birth cohort survives is the age x at which the life table function $\ell(x)$ is s. By definition, under the assumption $\mu(x) > 0$ for all $0 \leq x \leq \omega$, $a(s) = x$ if and only if $\ell(x) = s$. Equivalently, by definition, for every s in $0 \leq s \leq 1$ and every x in $0 \leq x \leq \omega$, $a(\ell(x)) = x$ and $\ell(a(s)) = s$. We define $a(1/2)$ as the median life length, that is, the age by which half the cohort has died.

Figure 2: **The function $\mathrm{x} = \mathrm{a}(\mathrm{s})$ that expresses the age x at which a fraction s of a birth cohort survives is the inverse of the life table function $\mathrm{s} = \ell(\mathrm{x})$ when the force of mortality is positive at every age. Apart from a reflection across the diagonal line $\mathrm{x} = \mathrm{s}$, the curve in this figure has the same relative shape as the curve in Figure 1 but the rescaling of both axes makes the two curves look different.**



For every s in $0 \leq s \leq 1$, we define the survival-specific force of mortality $\lambda(s)$ in terms of the age-specific force of mortality $\mu(x)$ in (1) in three equivalent ways:

$$(2) \qquad \lambda(s) = \mu(x) \text{ if } s = \ell(x); \quad \text{or} \quad \lambda(s) = \mu(a(s)); \quad \text{or} \quad \lambda(\ell(x)) = \mu(x).$$

In words, the survival-specific force of mortality $\lambda(s)$ at surviving proportion s equals the age-specific force of mortality $\mu(x)$ at the age x where the life table $\ell(x) = s$. The domain of the age-specific force of mortality μ is $0 \leq x \leq \omega$ while the domain of the survival-specific force of mortality λ is $0 \leq s \leq 1$. We give below an explicit formula (9) for the survival-specific force of mortality at surviving proportion s. This formula is analogous to (1) for the age-specific force of mortality.

The complete expectation of life at age x, $e(x)$, is the average number of years remaining to be lived by those who have attained age x:

$$e(x) = \frac{1}{\ell(x)} \int_{y=x}^{y=\omega-x} (y-x)\,\ell(y)\,\mu(y)\,dy. \tag{3}$$

Inserting the definition (1) in place of $\mu(y)$ in (3) and integrating by parts gives

$$e(x) = \frac{1}{\ell(x)} \int_{a=x}^{a=\omega} \ell(a)\,da, \tag{4}$$

a standard formula for life expectancy at age x (Keyfitz 1968:6).

1.2 Relationships

It is well known that the age-specific force of mortality $\mu(x)$ equals a constant $K > 0$ at every age x if and only if the life table is negative exponential with parameter K, i.e., $\ell(x) = \exp(-Kx)$. In this case, the expectation of life at age x is the reciprocal of the age-specific force of mortality:

$$e(x) = \frac{1}{K}. \tag{5}$$

From the definition (2) of the survival-specific force of mortality, it is evident that $\lambda(s) = K > 0$ at every surviving proportion s if and only if the life table is negative exponential with parameter K. Thus K in (5) may be viewed as a constant force of mortality, both age-specific and survival-specific.

Generalizations (6) and (8) extend (5) when the survival-specific force of mortality is *not* constant. These generalizations seem to be new.

A first generalization of (5) states that

$$e(x) = \frac{1}{\ell(x)} \int_{s=0}^{s=\ell(x)} \frac{ds}{\lambda(s)}. \tag{6}$$

In words, the expectation of life at age x is the average reciprocal of the survival-specific force of mortality weighted by the life-table deaths ds after age x. When $\lambda(s) = K$, (6) simplifies to (5) because

$$\frac{1}{\ell(x)} \int_{s=0}^{s=\ell(x)} ds = 1. \tag{7}$$

One can entirely eliminate age x (years of life lived in the past) from life expectancy (average years of life to be lived in the future) by defining a survival-specific life expectancy $E(s)$ (analogous to the survival-specific force of mortality defined above) as the life expectancy when the surviving proportion of the birth cohort is s. Thus by definition, $E(s) = e(x)$ if $s = \ell(x)$ and equivalently $E(s) = e(a(s))$ and $E(\ell(x)) = e(x)$.

A second generalization of (5) is to rewrite (6) as

$$E(s) = \frac{1}{s} \int_{s'=0}^{s'=s} \frac{ds'}{\lambda(s')}. \tag{8}$$

Here s' is the running variable for s. In words, the life expectancy when the surviving fraction is s is the death-weighted average of the reciprocal of the survival-specific force of mortality over each survival proportion smaller than s. The substantive difference between (6) and (8) is that age x appears as an argument on both sides of (6) and nowhere in (8). An illustration of (8) using United States data will be discussed in section 4. Applications.

We also demonstrate survival-specific forms of the force of mortality:

$$\lambda(s) = -\frac{1}{s}\frac{ds}{da} = -\frac{1}{s}\left(\frac{da}{ds}\right)^{-1}. \tag{9}$$

2. Proofs

Since $a(s)$ and $\ell(x)$ are inverse functions, elementary calculus shows that

$$\frac{d\ell}{dx} = \left(\frac{da}{ds}\right)^{-1} = \frac{ds}{da} \tag{10}$$

and that it is permissible under an integral to write

$$ds = \left(\frac{da}{ds}\right)^{-1} da. \tag{11}$$

Then, for $s = \ell(x)$ and running variables $s' = l(x')$ (and with the equalities numbered for subsequent explanation), we have

$$\begin{aligned} E(s) &\overset{1}{=} e(x) \\ &\overset{2}{=} \frac{1}{\ell(x)} \int\limits_{x'=x}^{x'=\omega} \frac{-1}{\mu(x')} \frac{d\ell(x')}{dx'}\, dx' \\ &\overset{3}{=} \frac{1}{s} \int\limits_{s'=0}^{s'=s} \frac{+1}{\lambda(s')} \left(\frac{da}{ds'}\right)^{-1} da \\ &\overset{4}{=} \frac{1}{s} \int\limits_{s'=0}^{s'=s} \frac{ds'}{\lambda(s')}. \end{aligned} \tag{12}$$

Equality 1 in (12) holds by definition of E. Equality 2 takes (4) and replaces the integrand $\ell(a)$ in (4) with the result of exchanging $\ell(x)$ and $\mu(x)$ in (1). Equality 3 uses the definitions $s = \ell(x)$ and $\lambda(s) = \mu(x)$ from (2), changes the minus one to plus one because of the reversal in the direction of integration, and uses (10) to replace one derivative with another. Finally, equality 4 uses (11) to "cancel" the differentials da. This proves (8), and using the definition $s = \ell(x)$ gives (6).

Finally, (9) follows immediately from the definitions (1) and (2) and the fact (10).

3. History and related results

I believe I was the first to state a special case of (6) in my first problem set dated 4 October 1971 for an undergraduate course on mathematical population models which I introduced at Harvard University (Biology 150). Assuming $\ell(0) = 1$, I asked the students to prove that

$$e(0) = \int\limits_{l=0}^{1} \frac{dl}{\lambda(l)}. \tag{13}$$

The text for the course was Nathan Keyfitz's then recent *Introduction to the Mathematics of Population* (1968). When Keyfitz began teaching at Harvard in the fall of 1972, I showed him (13) to find out if he had seen it before. He had not. I believe Keyfitz subsequently published (13), but not (6) or (8), as an exercise in one of his books. I cannot find the citation. To my knowledge, (6) and (8) and (9) have appeared nowhere before and no proof of (6), (8) or (13) has been published previously. The inequality (14) below also seems to be new.

4. Applications

4.1 Lower bound inequality

The expression (6) for $e(x)$ yields a new lower bound on life expectancy at age x. The reciprocal function that maps each positive real number x into $1/x$ is strictly convex. Therefore, Jensen's inequality for the average of a convex function applies to (6) and yields

$$(14) \qquad e(x) = \frac{1}{\ell(x)} \int_{l=0}^{l=\ell(x)} \frac{dl}{\lambda(l)} \geq \frac{1}{\frac{1}{\ell(x)} \int_{l=0}^{l=\ell(x)} \lambda(l)\, dl} = \frac{\ell(x)}{\int_{l=0}^{l=\ell(x)} \lambda(l)\, dl}.$$

This inequality is strict unless all deaths occur at a single age, that is, unless the life table is rectangular (in which case the age-specific force of mortality is not positive at ages less than ω), or unless the age-specific force of mortality is constant at all ages, in which case the survival-specific force of mortality is also constant. In words, the expectation of life at age x is at least as large as (and, apart from a rectangular life table or a constant force of mortality, is greater than) the reciprocal of the average survival-specific force of mortality weighted by life-table deaths dl at surviving proportions less than $\ell(x)$, that is, at ages above x.

4.2 Lower bound in the exponential distribution

The special case when the (age-specific or survival-specific) force of mortality is a constant $K > 0$ for $0 \leq x \leq \infty$ and $0 \leq s \leq 1$ verifies and illustrates (14). The left side of (14) is then $e(x) = 1/K$. The numerator of the right side of (14) is $\ell(x) = \exp(-Kx)$, and the denominator of the right side is

$$(15) \qquad K \int_{l=0}^{l=\ell(x)} dl = K\,\ell(x) = K\,\exp(-Kx).$$

Thus the right side of (14) equals $\ell(x)/(K\,\ell(x)) = 1/K$ and equality holds in (14), as expected in the case of a constant force of mortality.

4.3 Discrete actuarial approximations

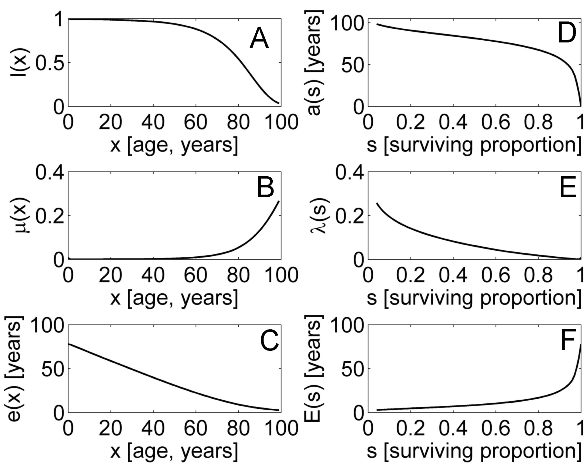
If we are given the life-table proportions ℓ_x surviving at each exact age x, the life-table probability q_x of dying by age $x + 1$ given survival to exact age x, and the expectation of life e_x at exact age x, then the estimation of $a(s)$, $\lambda(s)$, and $E(s)$ for given surviving proportions s requires only linear (or other) interpolation. For example, the Matlab command `interp1(lx,x,s,'spline')` uses piecewise cubic spline interpolation to produce $a(s)$ from three arguments: the life table expressed as a vector lx, a vector x of ages, and a vector s of proportions surviving. Similarly, Matlab command `interp1(lx,qx,s,'spline')` estimates the survival-specific force of mortality $\lambda(s)$ and Matlab command `interp1(lx,ex,s,'spline')` estimates the survival-specific life expectancy $E(s)$. Both spline and linear interpolation were tried and the results were very similar. Spline interpolation was preferred to linear interpolation because spline interpolation was smoother and took advantage of information outside the local interval of age.

The ability to compute $a(s)$, $\lambda(s)$, and $E(s)$ using existing actuarial methods plus interpolation is a double advantage. It requires little retooling of methods or software, and it sheds new light on, and raises fresh questions about, familiar data, as the following example is intended to show.

4.4 Example based on life tables of the United States in 2004

Arias (2007) tabulated ℓ_x, q_x, and e_x for exact ages $0, 1, 2, \ldots, 99$, and a terminal catch-all group 100 years or older, for the 2004 United States population and eight subpopulations. Figure 3 plots (A) $\ell(x) \approx \ell_x$, (B) $\mu(x) \approx q_x$, and (C) $e(x) \approx e_x$, for exact ages $x = 0, 1, \ldots, 99$, ignoring the final age group 100 years or older. Figure 3 also plots the corresponding (D) age function $a(s)$, (E) survival-specific force of mortality $\lambda(s)$, and (F) survival-specific life expectancy $E(s)$, as functions of the proportion surviving s, for $s = 0.04, 0.05, \ldots, 0.99, 1$. The values $s = 0.01, 0.02, 0.03$ are omitted because the fraction who survived to at least age 99 was $\ell_{99} = 0.03423$ and estimates of functions for arguments s smaller than ℓ_{99} would have required extrapolation rather than interpolation. Just as Arias's estimates for the age group 100 years or older required additional assumptions, estimates for $s < \ell_{99}$ would have required additional assumptions.

Figure 3: United States population in 2004 showing
(A) the life table $\ell(\mathrm{x})$,
(B) the age-specific force of mortality $\mu(\mathrm{x})$, and
(C) the expectation of remaining life $\mathrm{e}(\mathrm{x})$,
as functions of age x, based on Arias (2007:Table 1) and
(D) the age $\mathrm{a}(\mathrm{s})$ at which the proportion s survives,
(E) the survival-specific force of mortality $\lambda(\mathrm{s})$, and
(F) the expectation of remaining life $\mathrm{E}(\mathrm{s})$
as functions of the surviving proportion s.



The functions based on age differ from the corresponding functions based on the proportion surviving. The shoulder on the right of the age function (Figure 3D) is much more pronounced than that of the life table (Figure 3A), highlighting the rapid drop-off in age associated with the last increases in the proportion surviving from about 0.95 to 1. While the age-specific force of mortality (on this linear scale) (Figure 3B) rises notably only after the first six decades, and then rises gradually, the survival-specific force of mortality (Figure 3E) declines sharply for small values of s (corresponding to extreme old ages) and substantially across the entire range of s. Finally, while the age-specific expectation of life (Figure 3C) falls gradually, almost linearly, across the entire range of age, the survival-specific expectation of life Figure 3F) rises slowly until s approaches 1 and then rises quite sharply. Looked at another way, by moving s in a decreasing direction from right to left in Figure 3F, the greatest losses in expectation of remaining life occur when the first small fraction dies (at high s). Thereafter, as s decreases further, the decline in expectation of remaining life is much more gradual.

This perspective provides new ways to compare populations. To illustrate, Table 1 compares $E(1/2)$, the survival-specific life expectancy when half the cohort survives, as defined in (8), for the total population and eight subpopulations of the United States in 2004 (Figure 4), based on data of Arias (2007:Tables 1-9). For example, for the total population of the U.S. in 2004 (Arias 2007:Table 1), by exact age 81 the proportion surviving was 0.50987 with remaining life expectancy of 8.6 years and by exact age 82 the proportion surviving was 0.47940 with remaining life expectancy of 8.2 years. By linear interpolation I estimated a life expectancy of 8.4704 years when the proportion surviving was 0.5, and by spline interpolation I estimated a life expectancy of 8.4708, so I tabulated $E(1/2) = 8.5$ years. For simplicity in this illustration, the median life length $a(1/2)$, that is, the age at which the proportion surviving s equaled $1/2$, was approximated by the whole number of years of life completed, that is, by the integer part of $a(1/2)$. Males' $E(1/2)$ exceeded females' $E(1/2)$ by 0.8 year but half the male cohort had died by age 78, five years younger than half the female cohort had died, by age 83. Similarly, the black population's $E(1/2)$ exceeded the white population's $E(1/2)$ by 2.3 years but half the black cohort had died by age 76, five years younger than half the white cohort died, at age 81. Of the subpopulations considered in Table 1, black males had the longest $E(1/2)$, 10.9 years, and the shortest median life length, reaching half survival youngest, at 72 years. White females had the shortest $E(1/2)$, 7.7 years, and the longest median life length, 83 years, tied for oldest with all U.S. females. The younger half of a cohort died, that is, the shorter the median life length, the longer its life expectancy at that age.

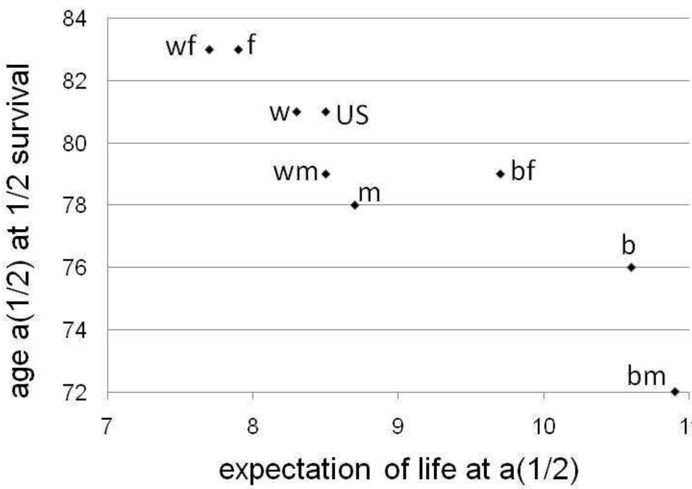
This relationship is the opposite of the relationship expected from the simplest model, when the force of mortality is a constant K and the life table is negative exponential $\ell(x) = \exp(-Kx)$. Then $a(1/2) = \ln(2)/K$ while the expectation of life (at any age or any surviving proportion) is $1/K$. Not surprisingly, the higher the force of mortality, the sooner half the cohort dies and the shorter the life expectancy. Both $a(1/2)$ and $E(1/2) = e(a(1/2))$ are inversely proportional to the force of mortality K and are directly proportional to one another in the family of negative exponential life tables with parameter K, contrary to the observations of United States subpopulations.

Understanding why U.S. subpopulations with shorter median life length had longer remaining expectation of life at that age is a topic for further theoretical and empirical analysis. Theoretically consider a family of life tables of specified form indexed by a parameter or several parameters (for example, the family of negative exponential life tables is indexed by the parameter K). One problem is to find necessary and sufficient conditions on the form of the life table and the values of the parameter(s) such that $E(s)$ and $a(s)$ are positively (or, negatively) associated as a parameter increases within some range. An empirical problem is to identify the demographic, economic, and cultural conditions under which $E(s)$ and $a(s)$ are observed to be positively (or, negatively) associated, and to interpret these conditions in terms of the theoretical conditions.

Table 1: Life expectancy $\mathbf{E(1/2)}$ and the year of age $\mathbf{x}$ to $\mathbf{x+1}$ in which half the life-table cohort survived, in the United States' 2004 total population and selected subsets

Population	Life expectancy $\mathbf{E(1/2)}$ when half the life-table cohort survived (years)	Age when half the life-table cohort survived (years)
Total population	8.5	81-82
Males	8.7	78-79
Females	7.9	83-84
White population	8.3	81-82
White males	8.5	79-80
White females	7.7	83-84
Black population	10.6	76-77
Black males	10.9	72-73
Black females	9.7	79-80

Figure 4: Integer part of the age a at which half a cohort survived (vertical axis) as a function of the complete expectation of life at age a, for the United States total population and eight subpopulations, 2004, estimated by interpolation from data of Arias (2007)



US = total population, w = white, b = black, m = male, f = female

5. Acknowledgements

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Due 4 October 1971

1. Prove that $\overset{\circ}{e}_0 = \int_{\ell_x=0}^{1} \frac{d\ell_x}{\mu_x}$
   Can you provide an intuitive argument why this is true?

2. If $\mu_x = \frac{x}{x+1}$, find ℓ_x. (Use tables of integrals if necessary.)

3. If $\mu_x = \frac{x}{x+c}$, $c \neq 0$, $c \neq 1$, find ℓ_x. (Try changing the scale from x to $t = x/c$, then use problem 2.)

4. Invent a model of mortality that leads to $\mu_x = x/(x+c)$.

5. In a cohort subject to a probability of death at each age x that is independent of age, $q_x = q$, prove that the number of deaths d_x at each age is given by

$$d_x = \ell_0 \frac{\ell_x}{\sum_0^\infty \ell_x} \qquad \text{(Deevey 1947)}$$

   Hence in a stationary population with mortality independent of age, deaths by age can be determined from age-composition of the population and vice versa. Application: M.M. Nice, Studies on the life history of the song sparrow. Vol 1. A population study of the song sparrow. Trans. Linn. Soc. NY, 4, 1937.

6. Chia-Tung Pan (Studies on the host-parasite relationship between *Schistosoma mansoni* and the snail *Australorbis glabratus*, *American Journal of Tropical Medicine and Hygiene* 14(6):931-976, 1965) selected two groups of snails of the same species having the same initial average size and infected one of the groups with parasites. The other was a control. He then observed the survivorship and fertility of the two groups. Note that the observations of survivorship begin long after the egg stage. Assume each egg mass laid contained 26 eggs. Observe interesting things in these data.

Week post infection	0	1	2	3	4	5	6	7	8	9	10	30
Infected snails surviving	150	150	147	144	142	141	140	135	126	116	97	2
Egg masses laid	0	2	18	64	22	1	0	0	0	0	0	0
Control snails surviving	100	100	99	99	98	97	97	97	96	96	95	60*
Egg masses laid	0	0	8	75	127	259	262	387	271	229	205	114*

*21 control snails were killed in a laboratory accident in week 22.