# Elementary inequalities that involve two nonnegative vectors or functions


**Joel E. Cohen[†‡], J. H. B. Kemperman[§], and Gheorghe Zbăganu[¶]**

[†]The Rockefeller University and Columbia University, 1230 York Avenue, Box 20, New York, NY 10021-6399; [§]Rutgers, The State University of New Jersey (Emeritus), 13 Sandra Road, East Brunswick, NJ 08816; and [¶]Centre for Mathematical Statistics, Casa Academiel, Calea 13 Septembrie Număr 13, RO-76100 Bucharest 5, Romania



Contributed by Joel E. Cohen, July 27, 2004



**We report 96 inequalities with common structure, all elementary to state but many not elementary to prove. If $n$ is a positive integer, $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are arbitrary vectors in $\Re_+^n = [0, \infty)^n$, and $\rho(m_{ij})$ is the spectral radius of an $n \times n$ matrix with elements $m_{ij}$, then, for example:**

$$\sum_{i,j} \min((a_ia_j), (b_ib_j)) \leq \sum_{i,j} \min((a_ib_j), (b_ia_j)),$$

$$\sum_{i,j} \max((a_i + a_j), (b_i + b_j)) \geq \sum_{i,j} \max((a_i + b_j), (b_i + a_j)),$$

$$\rho(\min((a_ia_j), (b_ib_j))) \leq \rho(\min((a_ib_j), (b_ia_j))),$$

$$\sum_{i,j} \min((a_ia_j), (b_ib_j))x_ix_j \leq \sum_{i,j} \min((a_ib_j), (b_ia_j))x_ix_j,$$

$$\text{for all real } x_i,\ i = 1, \ldots, n,$$

$$\int\int \log[(f(x) + f(y))(g(x) + g(y))]d\mu(x)d\mu(y)$$

$$\leq \int\int \log[(f(x) + g(y))(g(x) + f(y))]d\mu(x)d\mu(y).$$

**The second inequality is obtained from the first inequality by replacing min with max and × with + and by reversing the direction of the inequality. The third inequality is obtained from the first by replacing the summation by the spectral radius. The fourth inequality is obtained from the first by taking each summand as a coefficient in a quadratic form. The fifth inequality is obtained from the first by replacing both outer summations by products, min by ×, × by +, and the nonnegative vectors $a$ and $b$ by nonnegative measurable functions $f$ and $g$. The proofs of these inequalities are mysteriously diverse.**


This brief article presents a family of inequalities, all elementary to state but many not elementary to prove, and describes some applications in information theory and operations research. Ref. 1 gives proofs and counterexamples.

Zbăganu (2) proved that if $n$ is a positive integer and $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are arbitrary vectors in $\Re_+^n = [0, \infty)^n$, then

$$\sum_{i,j} \min((a_ia_j), (b_ib_j)) \leq \sum_{i,j} \min((a_ib_j), (b_ia_j)). \qquad [1]$$

This inequality has a very simple structure. Reading from left to right on each side of Inequality **1**, one first uses the operator $S$ = addition (summation), the operator $I$ = minimum, and finally the operator $P$ = multiplication (product). Zbăganu's inequality may be written as $SIP<$ and is 1 of 64 possible inequalities in which each of $S$, $I$, and $P$ in Zbăganu's inequality is replaced by each of $S$, $I$, and $P$, and $A$ = maximum. [At various points we use different notations for the minimum, so it is useful to be forewarned that, for any real $x$, $y$, $I(x, y) = \min(x, y) = x \wedge y$. Similarly for the maximum, $A(x, y) = \max(x, y) = x \vee y$.] Each of $A$, $I$, $S$, and $P$ can operate on finite sets of any size. Thus, for example, in the inequality $SAS$, the left $S$ is the sum of matrix elements, whereas the right $S$ is the sum of a pair of numbers. Of these 64 inequalities, all are true when $n = 2$, and 62 are true when $n > 2$.

To pose a more general question, each of the four operators $\{A, I, S, P\}$ could be replaced by commutative operators. Let $a$ and $b$ be arbitrary vectors in $\Re^n$ (possibly required to be nonnegative). Let $D$, $E$, and $F$ be commutative operators (with domain and range to be specified). Assuming compatibility of all operations specified, when is it true that, for all pairs $a$ and $b$,

$$D(E(F(a_i, a_j), F(b_i, b_j))) \leq D(E(F(a_i, b_j), F(a_j, b_i))) \qquad [2]$$

or else that

$$D(E(F(a_i, a_j), F(b_i, b_j))) \geq D(E(F(a_i, b_j), F(a_j, b_i)))? \qquad [3]$$

Typically, $F$ maps U × U into V (such as U = $\Re$ and V = $\Re^+$), whereas $E$ maps V × V into W (such as W = $\Re$ or W = $\Re^+$), and $D$ operates on $n \times n$ matrices with values in W. The range of $D$ is taken to be some partially ordered set, including possibly all $n \times n$ matrices with the Loewner ordering.

If valid, Inequality **2** is denoted by $DEF<$, and Inequality **3** is denoted by $DEF>$. Except for some equalities, at most one of $DEF<$ and $DEF>$ will be true. Which of these two has at least a chance to be true usually can be seen from the special case when all elements of $a$ equal one constant and all elements of $b$ equal another. When the inequality is true in general, the direction of the inequality usually is determined by this special case, so one may as well speak briefly of the inequality $DEF$.

Because $A$, $I$, $S$, and $P$ are all associative, $DEF$ is true with the equality sign when $E = F$. This observation proves 16 inequalities (four choices for $E = F$ and four choices for $D$).

These inequalities extend further. When $D$ is the spectral radius of the nonnegative matrix, we write $D = R$. When we replace the summation $D = S$ by a quadratic form, we write $D = Q$. Each of these two formal mutations of Inequality **1** leads to 16 additional conjectured inequalities, giving a total of 96 = 64 + 16 + 16 conjectured inequalities. All 96 are true when $n = 2$. When $n > 2$, the inequalities $PSI$, $SPA$, $QPA$, and $RPA$ are all false in general. For each inequality where $D = S$, we may replace the vector pairs $a$ and $b$ by pairs of real-valued functions (frequently limited to nonnegative real values) and the summation $D = S$ by an integral.

We believe that these inequalities represent an important class of inequalities. Despite our efforts, we have not found any universal type of proof. In view of the exceptional cases, such a universal proof may not exist. Alternatively, if there is a totally new algebraic structure behind many of our results, it might well lead to a better understanding of why some results of type $DEF$ are true and (a few) others are false.


[‡]To whom correspondence should be addressed. E-mail: cohen@rockefeller.edu.

## Applications

Zbăganu (2) considered a question in information theory: If one of two messages must be sent over a channel with only two input symbols, A and B, and with $n$ output symbols, 1, …, $n$, is the chance of error in transmission minimized by sending the first message as AA and the second message as BB or alternatively by sending the first message as AB and the second message as BA? Zbăganu proved that a lower risk of receiving the wrong message is achieved by coding the two messages by the pairs of symbols AA and BB than by the pairs of symbols AB and BA. This result is equivalent to Inequality **1**. If $a_i$ represents the probability that the input symbol A is received as the output symbol $i$ and $b_j$ represents the probability that the input symbol B is received as the output symbol $j$, and if the channel is memoryless so that errors in transmission affect output independently for each input symbol, then the matrix $(a_ia_j)$ is the joint probability distribution of output symbols $(i, j)$ when the input symbols are AA, and the matrix $(b_ib_j)$ is the joint probability distribution of output symbols $(i, j)$ when the input symbols are BB and similarly for the matrices $(a_ib_j)$ and $(b_ia_j)$. The left side of Inequality **1** measures the similarity between the matrices $(a_ia_j)$ and $(b_ib_j)$, because it takes the value 1 when the matrices are identical and takes the value 0 when the matrices have disjoint support (that is, the elements of one matrix are zero whenever the corresponding elements of the other matrix are positive). Similarly, the right side of Inequality **1** measures the similarity between the matrices $(a_ib_j)$ and $(b_ia_j)$. Inequality **1** shows that a lower risk of receiving the wrong message is achieved by coding the two messages by the pairs of symbols AA and BB than by the pairs of symbols AB and BA.

Generalizations of Inequality **1** were suggested by generalizations of matrix multiplication important in operations research, including manufacturing theory and routing theory (refs. 3–5 and references therein), which suggested replacing each of the three operations in Inequality **1** (addition, min, and multiplication) by each of the four operations, min, max, addition, and multiplication.

For example,

$$\sum_{i,j} \max((a_i + a_j), (b_i + b_j)) \geq \sum_{i,j} \max((a_i + b_j), (b_i + a_j)) \qquad [4]$$

is obtained from Inequality **1** by replacing min with max, and × with +, and by reversing the direction of the inequality. This formula has a natural interpretation in the design of a manufacturing process. Suppose a product has two necessary components, components 1 and 2. Suppose these components are manufactured in parallel. Each component requires a process of two steps, steps 1 and 2. Two machines called A and B can be arranged in one of two manufacturing configurations. In configuration I, component 1 passes through machine A in step 1 and again through machine A in step 2, whereas component 2 passes through machine B in both steps 1 and 2. In the alternative configuration II, component 1 passes through machine A in step 1 and through machine B in step 2, whereas component 2 passes through machine B in step 1 and through machine A in step 2. The product is completed when both components have completed both steps. Which manufacturing configuration, I or II, has a shorter average time to produce a product? The time that each machine requires to complete a step depends on the environment in the factory (for example, the temperature or the voltage). Let us suppose that at each step the environment may be in one of $n$ possible states, $i = 1, \ldots, n$, and that these states are equally likely and independent between steps 1 and 2 but identical for both machines at each step. If the environment is in state $i$ at step 1, machine A requires time $a_i$ and machine B requires time $b_i$ to complete step 1; exactly the same is true at step 2. Thus, if the environment is in state $i$ at step 1 and in state $j$ at step 2 (which will occur with probability $1/n^2$), and if component 1 passes through machine A at step 1 and through machine B at step 2 (as in configuration II), then the time required to make component 1 is $a_i + b_j$, the time required to make component 2 is $b_i + a_j$, and the time required to complete the product is $\max((a_i + b_j), (b_i + a_j))$. If both sides of Inequality **4** are multiplied by $1/n^2$, then the left side represents the average production time in configuration I, whereas the right side represents the average production time in configuration II. Inequality **4** shows that configuration II is preferable to configuration I, because it has shorter average production time. The assumption in this example that each state of the environment is equally likely can be replaced by arbitrary probabilities for each environmental state by using the extension to quadratic forms that is described below.

In another example, suppose a factory located at X has two suppliers of a hazardous raw material. These suppliers are located at V and Z. The raw material is trucked from V to W in 1 day, transferred to a fresh truck, and trucked from W to X in a second day; the raw material is likewise transferred from Z to Y in 1 day and then in a fresh truck from Y to X in a second day. The factory uses two trucking companies, A and B, and for legal reasons is obliged to use both companies every day. (The raw material is highly sensitive, and the government does not permit the factory to depend on a single trucker.) The factory can use plan I or II to ship the material. In plan I, company A operates from V to W and W to X, and company B operates from Z to Y and Y to X. In plan II, company A operates from V to W and Y to X, and company B operates from Z to Y and W to X. The capacity of the trucks operated by both companies depends on the road conditions, which are affected by weather, landslides, and forest fires. On any given day, both trucking companies experience the same road conditions. Suppose that under condition $i = 1, \ldots, n$, the maximum capacity of the trucks available from company A (or B) is $a_i$ tons (or $b_i$ tons). If conditions are in state $i$ on the first day and state $j$ on the second day, then under plan I company A can ship $\min(a_i, a_j)$ tons of the material from V to X and company B can ship $\min(b_i, b_j)$ tons from Z to X; thus, the factory in X can receive $\min(a_i, a_j) + \min(b_i, b_j)$ tons. Under plan II, if conditions are in state $i$ on the first day and state $j$ on the second day, then the factory can get $\min(a_i, b_j)$ tons of the material from V via W and $\min(b_i, a_j)$ tons from Z via Y; thus, the factory in X can receive $\min(a_i, b_j) + \min(b_i, a_j)$ tons. Under the worst combination of circumstances $(i, j)$, the factory can count on receiving $\min_{i,j}(\min(a_i, a_j) + \min(b_i, b_j))$ tons under plan I and $\min_{i,j}(\min(a_i, b_j) + \min(b_i, a_j))$ tons under plan II. Inequality $ISI>$ in Table 1 tells the factory that plan I assures at least as great a supply of the raw material as plan II. Inequality $ASI>$ shows that the maximum possible delivery under plan I is at least as great as that under plan II. If the $n$ conditions are equally likely and independent from one day to the next, then inequality $SSI>$ guarantees the company that plan I has at least as great an average delivery of the material as plan II. If condition $i$ occurs with probability $p_i$ and independently from day to day, then $QSI>$ guarantees that $\Sigma_{ij}(\min(a_i, a_j) + \min(b_i, b_j))p_ip_j \geq \Sigma_{ij}(\min(a_i, b_j) + \min(b_i, a_j))p_ip_j$, i.e., plan I has a better average delivery rate than plan II.

## Results

Table 1 states explicitly 48 of the 64 inequalities that involve only $S$, $P$, $I$, and $A$, along with some generalizations of these, including two inequalities, $PSI$ and $SPA$, identified as false. Table 1 omits the 16 true equalities $DEF$ where $E = F$. For each true inequality $SEF$ in Table 1, the corresponding inequalities

**Table 1. Inequalities of the form $DEF<$ or $DEF>$, where $D, E, F \in \{A, I, P, S\}$, excluding the 16 cases $DEF=$ when $E = F$**

| $DEF$ | Explicit form and generalizations (when possible) | Proof |
|---|---|---|
| $IIP<$ | $\wedge_{i,j}((a_ia_j) \wedge (b_ib_j)) \le \wedge_{i,j}((a_ib_j) \wedge (b_ia_j))$<br>$\wedge_{x,y}((f(x)f(y)) \wedge (g(x)g(y))) \le \wedge_{x,y}((f(x)g(y)) \wedge (f(y)g(x)))$ | Easy: section 3 |
| $IIS<$ | $\wedge_{i,j}((a_i + a_j) \wedge (b_i + b_j)) \le \wedge_{i,j}((a_i + b_j) \wedge (b_i + a_j))$<br>$\wedge_{x,y}((f(x) + f(y)) \wedge (g(x) + g(y))) \le \wedge_{x,y}((f(x) + g(y)) \wedge (f(y) + g(x)))$ | Easy: section 3 |
| $IIA<$ | $\wedge_{i,j}((a_i \vee a_j) \wedge (b_i \vee b_j)) \le \wedge_{i,j}((a_i \vee b_j) \wedge (b_i \vee a_j))$<br>$\wedge_{x,y}((f(x) \vee f(y)) \wedge (g(x) \vee g(y))) \le \wedge_{x,y}((f(x) \vee g(y)) \wedge (f(y) \vee g(x)))$ | Easy: section 3 |
| $IPI>$ | $\wedge_{i,j}((a_i \wedge a_j)(b_i \wedge b_j)) \ge \wedge_{i,j}((a_i \wedge b_j)(b_i \wedge a_j))$<br>$\wedge_{x,y}((f(x) \wedge f(y))(g(x) \wedge g(y))) \ge \wedge_{x,y}((f(x) \wedge g(y))(f(y) \wedge g(x)))$ | Easy: section 3 |
| $IPS<$ | $\wedge_{i,j}((a_i + a_j)(b_i + b_j)) \le \wedge_{i,j}((a_i + b_j)(b_i + a_j))$<br>$\wedge_{x,y}((f(x) + f(y))(g(x) + g(y))) \le \wedge_{x,y}((f(x) + g(y))(f(y) + g(x)))$ | Easy: section 3 |
| $IPA<$ | $\wedge_{i,j}((a_i \vee a_j)(b_i \vee b_j)) \le \wedge_{i,j}((a_i \vee b_j)(b_i \vee a_j))$<br>$\wedge_{x,y}((f(x) \vee f(y))(g(x) \vee g(y))) \le \wedge_{x,y}((f(x) \vee g(y))(f(y) \vee g(x)))$ | Easy: section 3 |
| $ISI>$ | $\wedge_{i,j}((a_i \wedge a_j) + (b_i \wedge b_j)) \ge \wedge_{i,j}((a_i \wedge b_j) + (b_i \wedge a_j))$<br>$\wedge_{x,y}((f(x) \wedge f(y)) + (g(x) \wedge g(y))) \ge \wedge_{x,y}((f(x) \wedge g(y)) + (f(y) \wedge g(x)))$ | Easy: section 3 |
| $ISP>$ | $\wedge_{i,j}((a_ia_j) + (b_ib_j)) \ge \wedge_{i,j}((a_ib_j) + (b_ia_j))$<br>$\wedge_{x,y}((f(x)f(y)) + (g(x)g(y))) \ge \wedge_{x,y}((f(x)g(y)) + (f(y)g(x)))$ | Easy: section 3 |
| $ISA<$ | $\wedge_{i,j}((a_i \vee a_j) + (b_i \vee b_j)) \le \wedge_{i,j}((a_i \vee b_j) + (b_i \vee a_j))$<br>$\wedge_{x,y}((f(x) \vee f(y)) + (g(x) \vee g(y))) \le \wedge_{x,y}((f(x) \vee g(y)) + (f(y) \vee g(x)))$ | Easy: section 3 |
| $IAI>$ | $\wedge_{i,j}((a_i \wedge a_j) \vee (b_i \wedge b_j)) \ge \wedge_{i,j}((a_i \wedge b_j) \vee (b_i \wedge a_j))$<br>$\wedge_{x,y}((f(x) \wedge f(y)) \vee (g(x) \wedge g(y))) \ge \wedge_{x,y}((f(x) \wedge g(y)) \vee (f(y) \wedge g(x)))$ | Easy: section 3 |
| $IAP>$ | $\wedge_{i,j}((a_ia_j) \vee (b_ib_j)) \ge \wedge_{i,j}((a_ib_j) \vee (b_ia_j))$<br>$\wedge_{x,y}((f(x)f(y)) \vee (g(x)g(y))) \ge \wedge_{x,y}((f(x)g(y)) \vee (f(y)g(x)))$ | Easy: section 3 |
| $IAS>$ | $\wedge_{i,j}((a_i + a_j) \vee (b_i + b_j)) \ge \wedge_{i,j}((a_i + b_j) \vee (b_i + a_j))$<br>$\wedge_{x,y}((f(x) + f(y)) \vee (g(x) + g(y))) \ge \wedge_{x,y}((f(x) + g(y)) \vee (f(y) + g(x)))$ | Easy: section 3 |
| | For the *PEF* inequalities, $\mu$ is a positive measure. | |
| $PIP<$ | $\Pi_{i,j}((a_ia_j) \wedge (b_ib_j)) \le \Pi_{i,j}((a_ib_j) \wedge (b_ia_j))$<br>$\int\int \log[(f(x)f(y)) \wedge (g(x)g(y))]d\mu(x)d\mu(y) \le \int\int \log[(f(x)g(y)) \wedge (g(x)f(y))]d\mu(x)d\mu(y)$<br>$E(\log(f(X)f(Y) \wedge g(X)g(Y))) \le E(\log(f(X)g(Y) \wedge g(X)f(Y)))$ | Is *SIS* |
| $PIS<$ | $\Pi_{i,j}((a_i + a_j) \wedge (b_i + b_j)) \le \Pi_{i,j}((a_i + b_j) \wedge (b_i + a_j))$<br>$\int\int \log[(f(x) + f(y)) \wedge (g(x) + g(y))]d\mu(x)d\mu(y) \le \int\int \log[(f(x) + g(y)) \wedge (g(x) + f(y))]d\mu(x)d\mu(y)$<br>$E(\log((f(X) + f(Y)) \wedge (g(X) + g(Y)))) \le E(\log((f(X) + g(Y)) \wedge (g(X) + f(Y))))$ | Theorem 6.11 |
| $PIA<$ | $\Pi_{i,j}((a_i \vee a_j) \wedge (b_i \vee b_j)) \le \Pi_{i,j}((a_i \vee b_j) \wedge (b_i \vee a_j))$<br>$\int\int \log[(f(x) \vee f(y)) \wedge (g(x) \vee g(y))]d\mu(x)d\mu(y) \le \int\int \log[(f(x) \vee g(y)) \wedge (g(x) \vee f(y))]d\mu(x)d\mu(y)$<br>$E(\log((f(X) \vee f(Y)) \wedge (g(X) \vee g(Y)))) \le E(\log((f(X) \vee g(Y)) \wedge (g(X) \vee f(Y))))$ | Is *SIA* |
| $PPI>$ | $\Pi_{i,j}((a_i \wedge a_j)(b_i \wedge b_j)) \ge \Pi_{i,j}((a_i \wedge b_j)(b_i \wedge a_j))$<br>$\int\int \log[(f(x) \wedge f(y))(g(x) \wedge g(y))]d\mu(x)d\mu(y) \ge \int\int \log[(f(x) \wedge g(y))(g(x) \wedge f(y))]d\mu(x)d\mu(y)$<br>$E(\log((f(X) \wedge f(Y))(g(X) \wedge g(Y)))) \ge E(\log((f(X) \wedge g(Y))(g(X) \wedge f(Y))))$ | Is *SSI* |
| $PPS<$ | $\Pi_{i,j}((a_i + a_j)(b_i + b_j)) \le \Pi_{i,j}((a_i + b_j)(b_i + a_j))$<br>$\int\int \log[(f(x) + f(y))(g(x) + g(y))]d\mu(x)d\mu(y) \le \int\int \log[(f(x) + g(y))(g(x) + f(y))]d\mu(x)d\mu(y)$<br>$E(\log((f(X) + f(Y))(g(X) + g(Y)))) \le E(\log((f(X) + g(Y))(g(X) + f(Y))))$ | Corollary 4.10 |
| $PPA<$ | $\Pi_{i,j}((a_i \vee a_j)(b_i \vee b_j)) \le \Pi_{i,j}((a_i \vee b_j)(b_i \vee a_j))$<br>$\int\int \log[(f(x) \vee f(y))(g(x) \vee g(y))]d\mu(x)d\mu(y) \le \int\int \log[(f(x) \vee g(y))(g(x) \vee f(y))]d\mu(x)d\mu(y)$<br>$E(\log((f(X) \vee f(Y))(g(X) \vee g(Y)))) \le E(\log((f(X) \vee g(Y))(g(X) \vee f(Y))))$ | Is *SSA* or<br>corollary 4.10 |
| $PSI>$ | $\Pi_{i,j}((a_i \wedge a_j) + (b_i \wedge b_j)) \ge \Pi_{i,j}((a_i \wedge b_j) + (b_i \wedge a_j))$ | False; true for $n = 2$ |
| $PSP>$ | $\Pi_{i,j}((a_ia_j) + (b_ib_j)) \ge \Pi_{i,j}((a_ib_j) + (b_ia_j))$<br>$\int\int \log[(f(x)f(y)) + (g(x)g(y))]d\mu(x)d\mu(y) \ge \int\int \log[(f(x)g(y)) + (g(x)f(y))]d\mu(x)d\mu(y)$<br>$E(\log(f(X)f(Y) + g(X)g(Y))) \ge E(\log(f(X)g(Y) + g(X)f(Y)))$ | Theorem 6.1 |
| $PSA<$ | $\Pi_{i,j}((a_i \vee a_j) + (b_i \vee b_j)) \le \Pi_{i,j}((a_i \vee b_j) + (b_i \vee a_j))$<br>$\int\int \log[(f(x) \vee f(y)) + (g(x) \vee g(y))]d\mu(x)d\mu(y) \le \int\int \log[(f(x) \vee g(y)) + (g(x) \vee f(y))]d\mu(x)d\mu(y)$<br>$E(\log(f(X) \vee f(Y) + g(X) \vee g(Y))) \le E(\log(f(X) \vee g(Y) + g(X) \vee f(Y)))$ | Implied by *GSA*<br>(corollary 5.9) |
| $PAI>$ | $\Pi_{i,j}((a_i \wedge a_j) \vee (b_i \wedge b_j)) \ge \Pi_{i,j}((a_i \wedge b_j) \vee (b_i \wedge a_j))$<br>$\int\int \log[(f(x) \wedge f(y)) \vee (g(x) \wedge g(y))]d\mu(x)d\mu(y) \ge \int\int \log[(f(x) \wedge g(y)) \vee (g(x) \wedge f(y))]d\mu(x)d\mu(y)$<br>$E(\log((f(X) \wedge f(Y)) \vee (g(X) \wedge g(Y)))) \ge E(\log((f(X) \wedge g(Y)) \vee (g(X) \wedge f(Y))))$ | Is *SAI* |
| $PAP>$ | $\Pi_{i,j}((a_ia_j) \vee (b_ib_j)) \ge \Pi_{i,j}((a_ib_j) \vee (b_ia_j))$<br>$\int\int \log[(f(x)f(y)) \vee (g(x)g(y))]d\mu(x)d\mu(y) \ge \int\int \log[(f(x)g(y)) \vee (g(x)f(y))]d\mu(x)d\mu(y)$<br>$E(\log((f(X)f(Y)) \vee (g(X)g(Y)))) \ge E(\log((f(X)g(Y)) \vee (g(X)f(Y))))$ | Is *SAS* |
| $PAS>$ | $\Pi_{i,j}((a_i + a_j) \vee (b_i + b_j)) \ge \Pi_{i,j}((a_i + b_j) \vee (b_i + a_j))$<br>$\int\int \log[(f(x) + f(y)) \vee (g(x) + g(y))]d\mu(x)d\mu(y) \ge \int\int \log[(f(x) + g(y)) \vee (g(x) + f(y))]d\mu(x)d\mu(y)$<br>$E(\log((f(X) + f(Y)) \vee (g(X) + g(Y)))) \ge E(\log((f(X) + g(Y)) \vee (g(X) + f(Y))))$ | Corollary 6.8 |
| | For the *SEF* inequalities, $\mu$ is a bounded signed measure and $x \in \Re^n$. | |
| $SIP<$ | $\Sigma_{i,j}((a_ia_j) \wedge (b_ib_j))x_ix_j \le \Sigma_{i,j}((a_ib_j) \wedge (b_ia_j))x_ix_j$<br>$\int\int[(f(x)f(y)) \wedge (g(x)g(y))]d\mu(x)d\mu(y) \le \int\int[(f(x)g(y)) \wedge (g(x)f(y))]d\mu(x)d\mu(y)$<br>$E(f(X)f(Y) \wedge g(X)g(Y)) \le E(f(X)g(Y) \wedge g(X)f(Y))$ | Theorem 5.3<br>Brownian bridge |

**Table 1. (continued)**

| *DEF* | Explicit form and generalizations (when possible) | Proof |
|---|---|---|
| *SIS*< | $\Sigma_{i,j}((a_i+a_j)\wedge(b_i+b_j))x_ix_j \le \Sigma_{i,j}((a_i+b_j)\wedge(b_i+a_j))x_ix_j$<br>$\iint[(f(x)+f(y))\wedge(g(x)+g(y))]d\mu(x)d\mu(y) \le \iint[(f(x)+g(y))\wedge(g(x)+f(y))]d\mu(x)d\mu(y)$<br>$E((f(X)+f(Y))\wedge(g(X)+g(Y))) \le E((f(X)+g(Y))\wedge(g(X)+f(Y)))$ | Theorem 5.5 |
| *SIA*< | $\Sigma_{i,j}((a_i\vee a_j)\wedge(b_i\vee b_j))x_ix_j \le \Sigma_{i,j}((a_i\vee b_j)\wedge(b_i\vee a_j))x_ix_j$<br>$\iint[(f(x)\vee f(y))\wedge(g(x)\vee g(y))]d\mu(x)d\mu(y) \le \iint[(f(x)\vee g(y))\wedge(g(x)\vee f(y))]d\mu(x)d\mu(y)$<br>$E((f(X)\vee f(Y))\wedge(g(X)\vee g(Y))) \le E((f(X)\vee g(Y))\wedge(g(X)\vee f(Y)))$ | Theorem 5.6 |
| *SPI*> | $\Sigma_{i,j}((a_i\wedge a_j)(b_i\wedge b_j))x_ix_j \ge \Sigma_{i,j}((a_i\wedge b_j)(b_i\wedge a_j))x_ix_j$<br>$\iint(f(x)\wedge f(y))(g(x)\wedge g(y))d\mu(x)d\mu(y) \ge \iint(f(x)\wedge g(y))(g(x)\wedge f(y))d\mu(x)d\mu(y)$<br>$E((f(X)\wedge f(Y))(g(X)\wedge g(Y))) \ge E((f(X)\wedge g(Y))(g(X)\wedge f(Y)))$ | Theorem 5.12; induction on $\|X\|$ |
| *SPS*< | $\Sigma_{i,j}((a_i+a_j)(b_i+b_j))x_ix_j \le \Sigma_{i,j}((a_i+b_j)(b_i+a_j))x_ix_j$<br>$\iint(f(x)+f(y))(g(x)+g(y))d\mu(x)d\mu(y) \le \iint(f(x)+g(y))(g(x)+f(y))d\mu(x)d\mu(y)$<br>$E((f(X)+f(Y))(g(X)+g(Y))) \le E((f(X)+g(Y))(g(X)+f(Y)))$ | Easy |
| *SPA*< | $\Sigma_{i,j}((a_i\vee a_j)(b_i\vee b_j)) \le \Sigma_{i,j}((a_i\vee b_j)(b_i\vee a_j))$ | False; true for $n = 2$ |
| *SSI*> | $\Sigma_{i,j}((a_i\wedge a_j)+(b_i\wedge b_j))x_ix_j \ge \Sigma_{i,j}((a_i\wedge b_j)+(b_i\wedge a_j))x_ix_j$<br>$\iint[(f(x)\wedge f(y))+(g(x)\wedge g(y))]d\mu(x)d\mu(y) \ge \iint[(f(x)\wedge g(y))+(g(x)\wedge f(y))]d\mu(x)d\mu(y)$<br>$E((f(X)\wedge f(Y))+(g(X)\wedge g(Y))) \ge E((f(X)\wedge g(Y))+(g(X)\wedge f(Y)))$ | Theorem 5.11 |
| *SSP*> | $\Sigma_{i,j}((a_ia_j)+(b_ib_j))x_ix_j \ge \Sigma_{i,j}((a_ib_j)+(b_ia_j))x_ix_j$<br>$\iint[(f(x)f(y))+(g(x)g(y))]d\mu(x)d\mu(y) \ge \iint[(f(x)g(y))+(g(x)f(y))]d\mu(x)d\mu(y)$<br>$E((f(X)f(Y))+(g(X)g(Y))) \ge E((f(X)g(Y))+(g(X)f(Y)))$ | Easy |
| *SSA*< | $\Sigma_{i,j}((a_i\vee a_j)+(b_i\vee b_j))x_ix_j \le \Sigma_{i,j}((a_i\vee b_j)+(b_i\vee a_j))x_ix_j$<br>$\iint[(f(x)\vee f(y))+(g(x)\vee g(y))]d\mu(x)d\mu(y) \le \iint[(f(x)\vee g(y))+(g(x)\vee f(y))]d\mu(x)d\mu(y)$<br>$E((f(X)\vee f(Y))+(g(X)\vee g(Y))) \le E((f(X)\vee g(Y))+(g(X)\vee f(Y)))$ | Theorem 5.7 |
| *SAI*> | $\Sigma_{i,j}((a_i\wedge a_j)\vee(b_i\wedge b_j))x_ix_j \ge \Sigma_{i,j}((a_i\wedge b_j)\vee(b_i\wedge a_j))x_ix_j$<br>$\iint[(f(x)\wedge f(y))\vee(g(x)\wedge g(y))]d\mu(x)d\mu(y) \ge \iint[(f(x)\wedge g(y))\vee(g(x)\wedge f(y))]d\mu(x)d\mu(y)$<br>$E((f(X)\wedge f(Y))\vee(g(X)\wedge g(Y))) \ge E((f(X)\wedge g(Y))\vee(g(X)\wedge f(Y)))$ | Implied by *SIA* |
| *SAP*> | $\Sigma_{i,j}((a_ia_j)\vee(b_ib_j))x_ix_j \ge \Sigma_{i,j}((a_ib_j)\vee(b_ia_j))x_ix_j$<br>$\iint(f(x)f(y)\vee g(x)g(y))d\mu(x)d\mu(y) \ge \iint(f(x)g(y)\vee g(x)f(y))d\mu(x)d\mu(y)$<br>$E((f(X)f(Y))\vee(g(X)g(Y))) \ge E((f(X)g(Y))\vee(g(X)f(Y)))$ | Corollary 5.4; implied by *SIP* |
| *SAS*> | $\Sigma_{i,j}((a_i+a_j)\vee(b_i+b_j))x_ix_j \ge \Sigma_{i,j}((a_i+b_j)\vee(b_i+a_j))x_ix_j$<br>$\iint[(f(x)\vee f(y))\vee(g(x)\vee g(y))]d\mu(x)d\mu(y) \ge \iint[(f(x)\vee g(y))\vee(g(x)\vee f(y))]d\mu(x)d\mu(y)$<br>$E((f(X)+f(Y))\vee(g(X)+g(Y))) \ge E((f(X)+g(Y))\vee(g(X)+f(Y)))$ | Implied by *SIS* |
| *AIP*< | $\bigvee_{i,j}((a_ia_j)\wedge(b_ib_j)) \le \bigvee_{i,j}((a_ib_j)\wedge(b_ia_j))$<br>$\bigvee_{x,y}((f(x)f(y))\wedge(g(x)g(y))) \le \bigvee_{x,y}((f(x)g(y))\wedge(f(y)g(x)))$ | Easy: section 3 |
| *AIS*< | $\bigvee_{i,j}((a_i+a_j)\wedge(b_i+b_j)) \le \bigvee_{i,j}((a_i+b_j)\wedge(b_i+a_j))$<br>$\bigvee_{x,y}((f(x)+f(y))\wedge(g(x)+g(y))) \le \bigvee_{x,y}((f(x)+g(y))\wedge(f(y)+g(x)))$ | Easy: section 3 |
| *AIA*< | $\bigvee_{i,j}((a_i\vee a_j)\wedge(b_i\vee b_j)) \le \bigvee_{i,j}((a_i\vee b_j)\wedge(b_i\vee a_j))$<br>$\bigvee_{x,y}((f(x)\vee f(y))\wedge(g(x)\vee g(y))) \le \bigvee_{x,y}((f(x)\vee g(y))\wedge(f(y)\vee g(x)))$ | Easy: section 3 |
| *API*> | $\bigvee_{i,j}((a_i\wedge a_j)(b_i\wedge b_j)) \ge \bigvee_{i,j}((a_i\wedge b_j)(b_i\wedge a_j))$<br>$\bigvee_{x,y}((f(x)\wedge f(y))(g(x)\wedge g(y))) \ge \bigvee_{x,y}((f(x)\wedge g(y))(f(y)\wedge g(x)))$ | Easy: section 3 |
| *APS*< | $\bigvee_{i,j}((a_i+a_j)(b_i+b_j)) \le \bigvee_{i,j}((a_i+b_j)(b_i+a_j))$<br>$\bigvee_{x,y}((f(x)+f(y))(g(x)+g(y))) \le \bigvee_{x,y}((f(x)+g(y))(f(y)+g(x)))$ | Easy: section 3 |
| *APA*< | $\bigvee_{i,j}((a_i\vee a_j)(b_i\vee b_j)) \le \bigvee_{i,j}((a_i\vee b_j)(b_i\vee a_j))$<br>$\bigvee_{x,y}((f(x)\vee f(y))(g(x)\vee g(y))) \le \bigvee_{x,y}((f(x)\vee g(y))(f(y)\vee g(x)))$ | Easy: section 3 |
| *ASI*> | $\bigvee_{i,j}((a_i\wedge a_j)+(b_i\wedge b_j)) \ge \bigvee_{i,j}((a_i\wedge b_j)+(b_i\wedge a_j))$<br>$\bigvee_{x,y}((f(x)\wedge f(y))+(g(x)\wedge g(y))) \ge \bigvee_{x,y}((f(x)\wedge g(y))+(f(y)\wedge g(x)))$ | Easy: section 3 |
| *ASP*> | $\bigvee_{i,j}((a_ia_j)+(b_ib_j)) \ge \bigvee_{i,j}((a_ib_j)+(b_ia_j))$<br>$\bigvee_{x,y}((f(x)f(y))+(g(x)g(y))) \ge \bigvee_{x,y}((f(x)g(y))+(f(y)g(x)))$ | Easy: section 3 |
| *ASA*< | $\bigvee_{i,j}((a_i\vee a_j)+(b_i\vee b_j)) \le \bigvee_{i,j}((a_i\vee b_j)+(b_i\vee a_j))$<br>$\bigvee_{x,y}((f(x)\vee f(y))+(g(x)\vee g(y))) \le \bigvee_{x,y}((f(x)\vee g(y))+(f(y)\vee g(x)))$ | Easy: section 3 |
| *AAI*> | $\bigvee_{i,j}((a_i\wedge a_j)\vee(b_i\wedge b_j)) \ge \bigvee_{i,j}((a_i\wedge b_j)\vee(b_i\wedge a_j))$<br>$\bigvee_{x,y}((f(x)\wedge f(y))\vee(g(x)\wedge g(y))) \ge \bigvee_{x,y}((f(x)\wedge g(y))\vee(f(y)\wedge g(x)))$ | Easy: section 3 |
| *AAP*> | $\bigvee_{i,j}((a_ia_j)\vee(b_ib_j)) \ge \bigvee_{i,j}((a_ib_j)\vee(b_ia_j))$<br>$\bigvee_{x,y}((f(x)f(y))\vee(g(x)g(y))) \ge \bigvee_{x,y}((f(x)g(y))\vee(f(y)g(x)))$ | Easy: section 3 |
| *AAS*> | $\bigvee_{i,j}((a_i+a_j)\vee(b_i+b_j)) \ge \bigvee_{i,j}((a_i+b_j)\vee(b_i+a_j))$<br>$\bigvee_{x,y}((f(x)+f(y))\vee(g(x)+g(y))) \ge \bigvee_{x,y}((f(x)+g(y))\vee(f(y)+g(x)))$ | Easy: section 3 |

Assume that $a > 0$, $b > 0$; in some cases, this condition can be relaxed. Assume $f$ and $g$ are measurable and nonnegative (or positive, where positivity is required for the expressions to make sense). In the right column, section, theorem, and corollary numbers refer to ref. 1, where proofs are given.

*REF* and *QEF* pertaining to the spectral radius and quadratic form are true for nonnegative $a, b \in \Re^n$. When *SEF* holds for all real (not merely nonnegative) $a, b \in \Re^n$, then *QEF* also holds for all real (not merely nonnegative) $a, b \in \Re^n$. The inequalities *PSI*, *SPA*, *QPA*, and *RPA* are all false in general. If $n = 2$, then all 96 inequalities are true.

Our inequalities yield a nice generalization of Inequality **1**, which is proven in ref. 1.

**Theorem.** *Let * be one of the four operations* $+$, $\times$, $\wedge$, *and* $\vee$ *on* $\Re$. *Let* $a, b \in \Re^n$. *Denote by* $a^*a$ *the* $n \times n$ *matrix* $a_{i,j} = a_i{}^*a_j$. *Then the matrix* $a^*a$ *is more different from* $b^*b$ *than* $a^*b$ *is from* $b^*a$. *Precisely, if* $\|A\| = \Sigma_{1\le i,j\le n} |a_{i,j}|$, *then*

$$\|a^*a - b^*b\| \ge \|a^*b - b^*a\|.$$

Another by-product of our inequalities was indicated to us by Victor de la Peña (personal communication). The inequality $SAS>$ may be written in terms of the independent and identically distributed random variables $X$ and $Y$ (which take their values in any measurable space that is the same for both), the real-valued functions $f$ and $g$, and the expectation operator $E$ (distinguishable by context from the earlier use of $E$ for an unspecified one of the binary operations $S$, $P$, $I$, and $A$) as

$$E((f(X) + f(Y)) \vee (g(X) + g(Y))) \ge E((f(X) + g(Y)) \vee (g(X) + f(Y))).$$

If we let $X$ and $Y$ be real-valued and $f(x) = x$, $g(x) = -x$, then

$$E((X + Y) \vee (-X - Y)) \ge E((X - Y) \vee (-X + Y))$$

or $E|X + Y| \ge E|X - Y|$. This is a special case of inequality 2.1 of Buja *et al*. (6) for independent and identically distributed scalar real-valued random variables (with $n = 1$ and $p = 1$ in their notation). By standard techniques, one then can prove for the Euclidean norm $\|.\|$ that $E\|X + Y\| \ge E\|X - Y\|$ for independent and identically distributed $n$-dimensional real random vectors $X$ and $Y$, because $\|x\|$ is an integral of $|\langle x, a\rangle|$, where $a$ belongs to the unit sphere. For related later results, see ref. 7.

## Open Problems

The main open problem is to derive more unified proofs of the inequalities.

A second major challenge is to complete our many partial results regarding the set $\Omega = \{(p, q, r) \in [-\infty, \infty]^3; E_pE_qE_r< \text{ or } E_pE_qE_r>\}$, where, for example, if $p, q, r \notin \{-\infty, 0, \infty\}$, we have $E_pE_qE_r<$ if and only if

$$\left(\sum_{1\le i,j\le n}\left((a_i^r + a_j^r)^{q/r} + (b_i^r + b_j^r)^{q/r}\right)^{p/q}\right)^{1/p} \le \left(\sum_{1\le i,j\le n}\left((a_i^r + b_j^r)^{q/r} + (b_i^r + a_j^r)^{q/r}\right)^{p/q}\right)^{1/p}$$

for all $n \ge 1$ and for all $a, b \in (0, \infty)^n$; similarly $E_pE_qE_r>$ if and only if $\le$ is replaced by $\ge$ in the inequality shown above. It is sufficient to investigate only three classes of inequalities: those with three-letter codes $IE_qE_r$, $PE_qE_r$, and $SE_qE_r$, where for a pair of positive numbers $x$ and $y$, $E_p(x, y) = (x^p + y^p)^{1/p}$, for $p \ne 0$. In particular, $E_1(x, y) = x + y$. Letting $p \to \infty$ leads to $E_\infty(x, y) = \max(x, y)$ and $p \to -\infty$ leads to $E_{-\infty}(x, y) = \min(x, y)$. In the collection $\{E_\alpha; \alpha \in \Re\backslash\{0\}\}$, the operator $P(x, y) = xy$ fits very nicely in place of the missing operator $E_0$. Roughly, $I = E_{-\infty}$, $H = E_{-1}$, $P = E_0$, $S = E_1$, and $A = E_\infty$, where $H$ stands for the harmonic operator $H(x, y) = 1/(1/x + 1/y)$. As part of this project, the monotonicity conjecture that if $0 < p \le q \le r$, then $E_pE_qE_r<$ holds remains to be proven. We proved this conjecture when $p = q$.

A third open problem is to prove the conjecture that if $a, b \in [0, \infty)^n$ and if inequality $SE_qE_r$ holds, then the corresponding $QE_qE_r$ holds, too.

Although we have completely analyzed a large number of inequalities (1), three mysteries remain. First, why should so many of these inequalities be true, given that they were conjectured by formal analogy? Second, why should the methods used to prove those conjectures that are true be so extraordinarily diverse? Third, what differentiates the few conjectures that turned out to be false from the overwhelming majority of others that turned out to be true?

J.E.C. and G.Z. dedicate this article to J.H.B.K. in celebration of his 80th birthday on July 16, 2004. We thank Ioan Cuculescu, Victor de la Peña, Daniel C. Reuman, and Radu Zaharopol for helpful comments. J.E.C. acknowledges with thanks the support of U.S. National Science Foundation Grant DEB9981552 and the hospitality of Mr. and Mrs. William T. Golden during this work.

1. Cohen, J. E., Kemperman, J. H. B. & Zbăganu, G. (2004) *Linear Algebra Appl*., in press.
2. Zbăganu, G. (2000) *Proc. Rom. Acad. Ser. A* **1,** 15–19.
3. Cuninghame-Green, R. (1979) *Minimax Algebra*, Lecture Notes in Economics and Mathematical Systems 166 (Springer, Heidelberg).
4. Cohen, J. E. (1988) *SIAM Rev*. **30,** 69–86.
5. Baccelli, F. & Hong, D. (2000) *Ann. Appl. Probab*. **10,** 779–827.
6. Buja, A., Logan, B. F., Reeds, J. A. & Shepp, L. A. (1994) *Ann. Stat*. **22,** 406–438.
7. Mattner, L. (2003) *Ann. Probab*. **31,** 914–925.