CIRCUITS SYSTEMS SIGNAL PROCESSING
VOL. 16, NO. 3, 1997, PP. 363–374

ORTHOGONAL CYCLE TRANSFORMS OF STOCHASTIC MATRICES*

S. Kalpazidou[1] *and Joel E. Cohen*[2]

Abstract. In this paper we investigate new Fourier series with respect to orthonormal families of directed cycles $\underline{\gamma}$, which occur in the graph of a recurrent stochastic matrix P. Specifically, it is proved that P may be approximated in a suitable Hilbert space by the Fourier series $\sum w_\gamma \underline{\gamma}$. This approach provides a proof in terms of Hilbert space of the cycle decomposition formula for finite stochastic matrices P.

1. Preliminaries

Let S be at most a denumerable set and let $P = (p_{ij}, i, j \in S)$ be an irreducible and positive-recurrent stochastic matrix. Then there exists a probability row-distribution $\pi = (\pi_i, i \in S)$ such that $\pi_i > 0, i \in S$, and

$$\sum_j \pi_i p_{ij} = \sum_j \pi_j p_{ji} , \quad i \in S . \tag{1}$$

From the point of view of homology theory, the "balance equations" (1) may be equivalently written as follows:

$$\pi_i p_{ij} = \sum_{c \in C} w_c J_c(i, j) , \quad i, j \in S , \tag{2}$$

where C is a collection of directed cycles (or circuits) c in S, the w_c's are positive numbers associated with c, and $J_c(i, j) = 1$ or 0 according to whether or not (i, j) is an edge of c. If, in addition, the cycle weights w_c are provided by a probabilistic algorithm involving the sample paths of the Markov chain on P, then equations (2) define a one-to-one transform $P \to (C, w_c)$, and $w_c \in C$, are called the probabilistic weights (see Theorem 4.1.1 of [4]). Often this probabilistic

* Received May 27, 1996; revised October 28, 1996.
[1] Aristotle University of Thessaloniki, Department of Mathematics, 54 006 Thessaloniki, Greece.
[2] Rockefeller University, Laboratory of Populations, 1230 York Avenue, Box 20, New York, New York 10021-6399; and Columbia University, New York, NY 10027.

algorithm is combined with a homologic one, in which case the corresponding cycle weights w_c are called the probabilistic-homologic weights (see Theorem 4.5.1 of [4]). Equations (2) are called the cycle representation formulas for the transition probabilities p_{ij}, $i, j \in S$.

As formal expressions, the cycle representation equations (2) lead to the question of whether or not these equations have a Hilbert space interpretation as decompositions on orthonormal collections. One approach to this question is given in the context of the theory of the unitary dilations of Riesz and Nagy (see [7] and [4]). Correspondingly, it is shown in [4] that the trigonometric functions may decompose P and the w_c into classic Fourier series.

On the other hand, the cycle representations (2) disclose certain homologic properties of P focused on the intrinsic structural interrelations between the edge weights $w(i, j) = \pi_i p_{ij}$, $i, j \in S$, and the cycle weights w_c, $c \in C$. This motivates the principal goal of this paper, which is to show the existence of a special orthonormal family $\Gamma = \{\underline{\gamma}_1, \underline{\gamma}_2, \dots\}$ of algebraic cycles connecting the ruling edge-cycle relations (2) with a Fourier representation for P. Namely, we shall show that πP may be viewed as a vector $\underline{w}$ in a suitable Hilbert space such that it is approximated by Fourier series on the orthonormal family Γ. Then, one original point of the paper is the definition of new Fourier series whose orthonormal bases are closer to the graph nature of P, or in general of any object that is characterized by a graph (1-complex).

A brief presentation of our exposition is as follows. We first define two Hilbert spaces $H(\mathcal{E})$ and $H(\mathcal{C})$ whose generators $\mathcal{E}$ and $\mathcal{C}$ are respectively described by using the collection E of directed edges and the collection C of directed cycles of the graph of P; that is, cl(span E) $= H(\mathcal{E})$, and cl(span C) $= H(\mathcal{C})$. Here the span $E = \mathcal{E}$ and the span $C = \mathcal{C}$ mean, respectively, the vector space generated by the algebraic chains associated with the directed edges of E and with the directed cycles of C, and cl denotes the topological closure.

As the algebraic cycles of $\mathcal{C}$ are not necessarily linearly independent, we shall replace $\mathcal{C}$ by a collection $\Gamma = \{\underline{\gamma}_1, \underline{\gamma}_2, \dots\}$ of linearly independent algebraic cycles such that the corresponding Hilbert space $H(\Gamma)$ is identical to $H(\mathcal{C})$. Furthermore, because $\mathcal{C}$ is an inner subspace of $H(\mathcal{E})$, we have

$$H(\mathcal{E}) = \mathrm{cl}\, \mathcal{C} \oplus \mathcal{C}\perp,$$

where cl $\mathcal{C}$ denotes the topologic closure of $\mathcal{C}$ in $H(\mathcal{E})$, $\mathcal{C}\perp$ is the orthogonal complement of $\mathcal{C}$, and "=" is understood as an isomorphism. Accordingly, the orthogonal projection on $\mathcal{C}$ will transform any vector $\underline{w}$ in $H(\mathcal{E})$ into a limit of Fourier series with respect to the orthonormal basis Γ; that is,

$$\underline{w} = \lim_{n\to\infty} \sum_{k=1}^{m(n)} \tilde{w}_k(n)\underline{\gamma}_k + \underline{u}, \tag{3}$$

where $\tilde{w}_k(n)$, $k = 1, 2, \dots$, and $m(n)$ denote the Fourier coefficients and length of summation of the corresponding approximating Fourier series, and $\underline{u} \in \mathcal{C}\perp$.

On the other hand, given P, we may consider the Fourier series

$$\underline{w}(n) = \sum_k \hat{w}_{\gamma_k}(n)\underline{\gamma}_k, \quad n = 1, 2, \ldots,$$

of the partial sums

$$\sum_{k=1}^{n} w_k \underline{c}_k, \quad n = 1, 2, \ldots,$$

provided by the cycle representations (2). Then, if we view πP as a vector $\underline{w}$ in $H(\mathcal{E})$, we have

$$\underline{w} = \lim_{n\to\infty} \sum_{k=1}^{m(n)} \hat{w}_{\gamma_k}(n)\underline{\gamma}_k, \tag{4}$$

where the convergence is understood in $H(\mathcal{E})$, and each $\hat{w}_{\gamma_k}(n)$ is completely defined by the cycle weights $w_{c_1}, \ldots, w_{c_n}$.

The presentation of the paper is as follows. In Section 2 we state the preceding arguments for finite irreducible stochastic matrices P and prove that when the Betti cycles $\underline{\gamma}_1, \ldots, \underline{\gamma}_B$ are provided by the graph of P, then they form an orthonormal basis and the corresponding probabilistic-homologic cycle representation formula provided by Theorem 4.5.1 of [3],

$$\underline{w} = \sum_{k=1}^{B} w_{\gamma_k}\underline{\gamma}_k, \tag{5}$$

coincides with the Fourier representation on $\underline{\gamma}_1, \ldots, \underline{\gamma}_B$. In Section 3 we investigate the case of stochastic matrices on a denumerable set of indices and prove formulas (3) and (4).

As the orthonormal cycles $\{\underline{\gamma}_k\}$ may also be viewed as discrete periodic functions with different periods, the corresponding Fourier transform $\sum w_{\gamma_k}\underline{\gamma}_k$ on cycles $\underline{\gamma}_k$ would provide good reasons of comparison with classic trigonometric series (see [8]). However, the special homologic nature of the graph of the transition probability functions discloses the existence of peculiar Fourier transforms on cycles, which are different from the classic Fourier transforms on the trigonometric functions.

2. Orthogonal cycle transforms for finite stochastic matrices

Let $S = \{1, 2, \ldots, n\}$, $n > 1$, and let $P = (p_{ij}, i, j = 1, 2, \ldots, n)$ be an irreducible stochastic matrix whose probability row-distribution is $\pi = (\pi_i, i = 1, \ldots, n)$. Let $G = G(P) = (S, E)$ be the oriented graph attached to P, where $E = \{b_1, \ldots, b_\tau\}$ denotes the set of directed edges endowed with an ordering. The orientation of G means that each edge b_k is an ordered pair (i, j) of points of S such that $p_{ij} > 0$, where i is the initial point and j is the endpoint. Sometimes we shall prefer the symbol $b_{(i,j)}$ for b_k when we need to point out the terminal points.

The irreducibility of P means that the graph G is connected; that is, for any pair (i, j) of states there exists a sequence $b_{(i,i_1)}, b_{(i_1,i_2)}, \ldots, b_{(i_s,j)}$ of edges of G connecting i to j. When $i = j$, then such a sequence is called a directed circuit of G. Usually a directed circuit c may also be determined by specifying the sequence of consecutive points; that is, $c = (i, i_1, i_2, \ldots, i_s, i)$. The number $s + 1$ is called the period of c. Throughout this paper we shall consider directed circuits $c = (i, i_1, i_2, \ldots, i_s, i)$, where the points $i, i_1, i_2, \ldots, i_s$ are all distinct.

Let C denote the collection of all directed circuits of G. Then according to [2]–[4] and [6] the matrix P is decomposed by the circuits $c \in C$ as follows:

$$\pi_i p_{ij} = \sum_{c \in C} w_c J_c(i, j) \, , \tag{6}$$

where any w_c is uniquely defined by a probabilistic algorithm, and J_c is the passage matrix of c introduced in the previous section. Furthermore, equations (6) are independent of the ordering of C.

Now we shall look for a suitable Hilbert space where the cycle decomposition (6) is equivalent with a Fourier-type decomposition for P, as explained in the first section. In this direction we shall consider two vector spaces C_0 and C_1 generated by the collections S and E, respectively (see [5] and [4]). Then any two elements $\underline{c}_0 \in C_0$ and $\underline{c}_1 \in C_1$ have the following expressions:

$$\underline{c}_0 = \sum_{h=1}^{n} x_h n_h = \underline{x}'\underline{n} \, , \quad x_h \in R \, , \; n_h \in S \, ,$$

$$\underline{c}_1 = \sum_{k=1}^{\tau} y_k b_k = \underline{y}'\underline{b} \, , \quad y_k \in R \, , \; b_k \in E \, ,$$

where R denotes the sets of reals. The elements of C_0 and C_1 are called, respectively, the zero-chains and the one-chains associated with the graph G.

Let $\delta : C_1 \to C_0$ be the boundary linear transformation defined as

$$\delta \underline{c}_1 = \underline{y}' \eta \underline{n} \, ,$$

where

$$\eta_{b_j n_s} \begin{array}{ll} = +1 \, , & \text{if } n_s \text{ is the endpoint of the edge } b_j; \\ = -1 \, , & \text{if } n_s \text{ is the initial point of the edge } b_j; \\ = 0 \, , & \text{otherwise} \, . \end{array}$$

Let

$$Z = \ker \delta = \{\underline{z} \in C_1 : \underline{z}' \eta = \underline{0}\}$$

where $\underline{0}$ is the neutral element of C_1. Then Z is a linear subspace of C_1, and the elements of Z are called one-cycles. One subset of Z is given by all the elements $\underline{c} = b_{i_1} + \cdots + b_{i_k} \in C_1$ whose edges $b_{i_1}, \ldots, b_{i_k}$ form a directed circuit c in the graph G. In general, the circuits occurring in the decomposition (6) of P determine linearly dependent one-cycles in Z. It is proved in [5] (see also [4]) that there are B one-cycles $\underline{\gamma}_1, \ldots, \underline{\gamma}_B$, which form a base for the linear subspace Z. The number B is called the Betti number of G, and $\underline{\gamma}_1, \ldots, \underline{\gamma}_B$ are called the Betti one-cycles of C_1 and may be defined by algebraic constructions on the polygonal lines of the graph G (see [4], p. 55). When $\underline{\gamma}_1, \ldots, \underline{\gamma}_B$ are induced by genuine directed circuits $\gamma_1, \ldots, \gamma_B$ of the graph G, then we call $\gamma_1, \ldots, \gamma_B$ the Betti circuits of G.

With these preparations we now prove

Lemma 1. *The vector space $Z = \ker \delta$ of one-cycles is a Hilbert space whose dimension is the Betti number of the graph.*

Proof. Let $\Gamma = \{\underline{\gamma}_1, \ldots, \underline{\gamma}_B\}$ be the set of Betti one-cycles of G, endowed with an ordering. Then

$$Z = \left\{ \sum_{k=1}^{B} a_k \underline{\gamma}_k \, , \; a_k \in R \right\} .$$

Define the mapping $\langle \, , \rangle : Z \times Z \to R$ as follows:

$$\left\langle \sum_{k=1}^{B} a_k \underline{\gamma}_k, \sum_{k=1}^{B} b_k \underline{\gamma}_k \right\rangle = \sum_{k=1}^{B} a_k b_k \, .$$

Then $\langle \, , \rangle$ is an inner product on Z, and consequently Z is metrizable with respect to the metric

$$d\left(\sum_{k=1}^{B} a_k \underline{\gamma}_k, \sum_{k=1}^{B} b_k \underline{\gamma}_k \right) = \sqrt{\sum_{k=1}^{B} (a_k - b_k)^2} \, .$$

Therefore $(Z, \langle \, , \rangle)$ is an inner product space where Γ is an orthonormal base. Accordingly, to any one-cycle $\underline{z} = \sum_{k=1}^{B} a_k \underline{\gamma}_k$ there correspond the Fourier coefficients $a_k = \left\langle \underline{z}, \underline{\gamma}_k \right\rangle$, $k = 1, \ldots, B$, with respect to the orthonormal base Γ.

Define the mapping $f : Z \to R^B$ as follows:

$$f\left(\sum_{k=1}^{B} a_k \underline{\gamma}_k \right) = (a_1, \ldots, a_B) \, .$$

Then f preserves inner-product-space structures; that is, f is a linear bijection that preserves inner products. In particular, f is an isometry. Then $(Z, \langle \, , \rangle)$ is a Hilbert space, whose dimension is B. The proof is complete. □

The previous result characterizes any finite connected graph G. Now we shall focus on graphs $G(P)$ associated with irreducible stochastic matrices P. Denote by B the Betti number of $G(P)$. Consider the collection C of cycles occurring in the decomposition (6), endowed with an ordering; that is, $C = \{c_1, \ldots, c_s\}, s > 0$. Then we have

Theorem 2. *Let $P = (p_{ij}, i, j = 1, \ldots, n)$ be an irreducible stochastic matrix whose invariant probability distribution is $\pi = (\pi_1, \ldots, \pi_n)$. Assume that the graph $G(P)$ contains a collection $\{\gamma_1, \ldots, \gamma_B\}$ of Betti circuits. Then πP has a*

Fourier representation with respect to $\Gamma = \{\underline{\gamma}_1, \ldots, \underline{\gamma}_B\}$, *where the Fourier coefficients are identical with the probabilistic-homologic cycle weights* $w_{\gamma_1}, \ldots, w_{\gamma_B}$; *that is,*

$$\sum_{(i,j)} \pi_i p_{ij} b_{(i,j)} = \sum_{k=1}^{B} w_{\gamma_k} \underline{\gamma}_k \,, \quad w_{\gamma_k} \in R \,, \tag{7}$$

with

$$w_{\gamma_k} = \left\langle \pi P, \underline{\gamma}_k \right\rangle \,, \quad k = 1, \ldots, B \,.$$

In terms of the (i, j)*-coordinate, equations (7) are equivalent to*

$$\pi_i p_{ij} = \sum_{k=1}^{B} w_{\gamma_k} J_{\gamma_k}(i, j) \,, \quad w_{\gamma_k} \in R; i, j \in S \,. \tag{8}$$

If P *is a recurrent stochastic matrix, then a similar representation (7) holds, except for a constant, on each recurrent class.*

Proof. Denote $w(i, j) = \pi_i p_{ij}$, $i, j = 1, \ldots, n$. Then πP may be viewed as a one-chain $\underline{w} = \sum_{(i,j)} w(i, j) b_{(i,j)}$. Because πP is balanced, $\underline{w}$ is a one-cycle; that is, $\underline{w} \in Z = \ker \delta$. Then, according to Lemma 1, $\underline{w}$ may be written as a Fourier series with respect to an orthonormal base $\Gamma = \{\underline{\gamma}_1, \ldots, \underline{\gamma}_B\}$ of Betti circuits of G; that is,

$$\underline{w} = \sum_{k=1}^{B} \left\langle \underline{w}, \underline{\gamma}_k \right\rangle \underline{\gamma}_k \,, \tag{9}$$

where $\left\langle \underline{w}, \underline{\gamma}_k \right\rangle$, $k = 1, \ldots, B$, are the corresponding Fourier coefficients.

On the other hand, the homologic cycle formula proved by Theorem 4.5.1 of [4] asserts that $\underline{w}$ may be written as

$$\underline{w} = \sum_{k=1}^{B} w_{\gamma_k} \underline{\gamma}_k \,, \tag{10}$$

where w_{γ_k}, $k = 1, \ldots, B$, are the probabilistic-homologic cycle weights given by a linear transformation of the probabilistic weights w_c, $c \in C$, occurring in (6); that is,

$$w_{\gamma_k} = \sum_{c \in C} A(c, \underline{\gamma}_k) w_c \,, \quad A(c, \underline{\gamma}_k) \in \mathbf{Z} \,,$$

where $\mathbf{Z}$ denotes the set of integers. As representation (10) is unique, we obtain that the Fourier representation (9) coincides with the homologic one (10); that is,

$$w_{\gamma_k} = \left\langle \underline{w}, \underline{\gamma}_k \right\rangle \,, \quad k = 1, 2, \ldots, B \,.$$

Accordingly, because $\underline{c} = \sum_k A(c, \underline{\gamma}_k) \underline{\gamma}_k$, then

$$A(c, \underline{\gamma}_k) = \left\langle \underline{c}, \underline{\gamma}_k \right\rangle \,, \quad k = 1, \ldots, B \,,$$

and therefore

$$w_{\gamma_k} = \sum_{c \in C} \left\langle \underline{c}, \underline{\gamma}_k \right\rangle w_c \, . \tag{11}$$

Let us now suppose that P has more than one recurrent class $\mathbf{e}$ in $S = \{1, \ldots, n\}$. Then we may apply the previous reasonings to each recurrent class $\mathbf{e}$ and to each balanced expression

$$\pi_e(i) p_{ij} = \sum_{k=1}^{B} w_{\gamma_k} J_{\gamma_k}(i, j) \, , \quad i, j \in \mathbf{e} \, ,$$

where $B = B_{\mathbf{e}}$ is the Betti number of the connected component of the graph $G(P)$ corresponding to $\mathbf{e}$, and $\pi_{\mathbf{e}} = \{\pi_{\mathbf{e}}(i)\}$ (with $\pi_{\mathbf{e}}(i) > 0$, for $i \in \mathbf{e}$, and $\pi_{\mathbf{e}}(i) = 0$ outside $\mathbf{e}$) is the invariant probability distribution associated to each recurrent class $\mathbf{e}$. The proof is complete. □

Remark. Let $w = (w(k), k = 1, 2, \ldots, B)$ be defined as

$$w(k) = w_{\gamma_k} \, , \quad k = 1, \ldots, B \, ,$$

where w_{γ_k}, $k = 1, \ldots, B$ are the probabilistic-homologic weights occurring in (10). Then the equation

$$w(k) = \sum_{c \in C} \left\langle \underline{c}, \underline{\gamma}_k \right\rangle w_c$$

may be interpreted as the inverse Fourier transform of the probabilistic weight function w_c, $c \in C$, associated with P.

3. Orthogonal cycle transforms of denumerable stochastic matrices

3.1.

Let S be a denumerable set and let $P = (p_{ij}, i, j \in S)$ be an irreducible and positive-recurrent stochastic matrix on S. Then considering an arbitrary ordering on S, there exists a probability row-distribution $\pi = (\pi_i, i \in S)$ such that $\pi_i > 0$, $i \in S$, and $\pi P = \pi$. As we have already mentioned, there exist algorithms that associate P with a collection $(C^*, \{w_c, c \in C^*\})$, where C^* denotes a denumerable collection of directed circuits c in S, and the w_c are positive numbers associated with c. Furthermore, if we choose an arbitrary ordering on C^*, say $C^* = \{c_1, c_2, \ldots, \}$, we have

$$\pi_i p_{ij} = \sum_{k=1}^{\infty} w_{c_k} J_{c_k}(i, j) \, , \quad i, j \in S \, , \tag{12}$$

where $J_{c_k}(i, j)$ is the passage function associated with c_k defined in Section 1.

Throughout our further presentation, in order to ensure the metrizability of the vector spaces associated with the graph (1-complex) on P, we shall assume that

P satisfies the local-finiteness condition. That is, for each $i \in S$ there are finitely many $j \in S$ such that $p_{ij} > 0$ or $p_{ji} > 0$ (see Proposition 1.10.8, p. 47, and Theorem 1.10.10, p. 48, of [1]).

Denote by (S, E) the oriented graph associated with P and consider an arbitrary, but fixed, ordering on each set S, E, and C^*. As in the finite case, the edges will be symbolized either by b_k, $k = 1, 2, \dots$, or by $b_{(i,j)}$, $i, j \in S$. Now we shall define two Hilbert spaces from the sets E and C^*. In this direction we first consider the sets

$$\mathcal{E} = \left\{ \underline{b} = \sum_{k=1}^{n} a_k b_k, n \in N, a_k \in R, b_k \in E \right\} ,$$

$$\mathcal{C} = \left\{ \underline{c} = \sum_{k=1}^{m} x_k \underline{c}_k, m \in N, x_k \in R, c_k \in C^* \right\} ,$$

where N is the set of positive integers, and the notation $\underline{c}_k$ will be explained shortly. The set $\mathcal{E}$ becomes a real vector space with respect to the operations $+$ and scalar multiplicity defined as:

$$\sum_{k=1}^{n} \mathrm{a}_k b_k + \sum_{k=1}^{m} \alpha_k b_k = \sum (\mathrm{a}_k + \alpha_k) b_k ,$$

$$\lambda \sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} \lambda a_k b_k , \quad \lambda \in R .$$

In an analogous way we organize the set $\mathcal{C}$ as a real vector space. Furthermore, as any directed circuit $c \in C^*$ may be associated with a vector $\underline{c} = \sum_{(i,j)} J_c(i, j) b_{(i,j)}$ in $\mathcal{E}$, then $\mathcal{C}$ is a vector subspace of $\mathcal{E}$. This motivates the notation $\underline{c}_k$ in the linear combinations of $\mathcal{C}$.

Now let us choose by Zorn's lemma a linearly independent subcollection Γ of C^* that generates $\mathcal{C}$. Define $\langle \, , \rangle_1 : \mathcal{E} \times \mathcal{E} \to R$ by

$$\left\langle \sum_{k=1}^{n} \mathrm{a}_k b_k, \sum_{k=1}^{m} \alpha_k b_k \right\rangle_1 = \sum_{k=1}^{\min(n,m)} \mathrm{a}_k \alpha_k .$$

Then $(\mathcal{E}, \langle \, , \rangle_1)$ is an inner product space. Analogously, define $\langle \, , \rangle_2 : \mathcal{C} \times \mathcal{C} \to R$ such that $(\mathcal{C}, \langle \, , \rangle_2)$ becomes an inner product space as well. Consequently, $\langle \, , \rangle_1$ and $\langle \, , \rangle_2$ induce the norms $| \; |_1$ and $| \; |_2$ on $\mathcal{E}$ and $\mathcal{C}$, respectively.

Because $\mathcal{E}$ and $\mathcal{C}$ are incomplete metric spaces, we shall further consider their completions $H(\mathcal{E})$ and $H(\mathcal{C})$ along with the corresponding extensions of $\langle \, , \rangle_1$ and $\langle \, , \rangle_2$. This means that $\mathcal{E}$ and $\mathcal{C}$ are isomorphic with certain inner product subspaces $H_0(\mathcal{E})$ and $H_0(\mathcal{C})$ such that cl $H_0(\mathcal{E}) = H(\mathcal{E})$ and cl $H_0(\mathcal{C}) = H(\mathcal{C})$, where cl symbolizes the topological closure with respect to the corresponding inner product topology. Furthermore, the sets $E = \{b_1, b_2, \dots\}$ and $\Gamma = \{\underline{\gamma}_1, \underline{\gamma}_2, \dots\}$ are orthonormal bases of $H(\mathcal{E})$ and $H(\mathcal{C})$, respectively, and consequently any

$\underline{x} \in H(\mathcal{E})$ and any $\underline{y} \in H(\mathcal{C})$ may be uniquely written as Fourier series

$$\underline{x} = \sum_{k=1}^{\infty} \mathrm{a}_k b_k ,$$

$$\underline{y} = \sum_{k=1}^{\infty} \alpha_k \gamma_k ,$$

where $\mathrm{a}_k = \langle \underline{x}, b_k \rangle$ and $\alpha_k = \left\langle \underline{y}, \underline{c}_k \right\rangle$, $k = 1, 2, \ldots$. Furthermore, the Riesz–Fischer representation theorem allows us to write

$$H(\mathcal{E}) = \left\{ \underline{x} = \sum_{k=1}^{\infty} \mathrm{a}_k b_k : \mathrm{a}_k \in R, \sum_{k=1}^{\infty} \mathrm{a}_k^2 < \infty \right\}$$

and

$$H(\mathcal{C}) = \left\{ \underline{y} = \sum_{k=1}^{\infty} \alpha_k \underline{\gamma}_k : \alpha_k \in R, \sum_{k=1}^{\infty} \alpha_k^2 < \infty \right\} .$$

Then the topological closure cl $\mathcal{C}$ of $\mathcal{C}$ in $H(\mathcal{E})$ defines an orthogonal projection from $H(\mathcal{E})$ to cl $\mathcal{C}$. That is, for any element $\underline{y} \in H(\mathcal{E})$ there exist a unique element $\underline{y}_1$ in cl $\mathcal{C}$ and $\underline{u} \in \mathcal{C}\perp$ such that

$$\underline{y} = \underline{y}_1 + \underline{u} . \tag{13}$$

3.2.

Now we shall turn back to our original stochastic matrix P and investigate the relation between πP and the Hilbert spaces $H(\mathcal{E})$ and $H(\mathcal{C})$. Recall that $\mathcal{C}$ is an inner product subspace of $H(\mathcal{E})$. Denote $w(i, j) = \pi_i p_{ij}$, $i, j \in S$ and define

$$\underline{w} = \sum_{(i,j)} w(i, j) b_{(i,j)} . \tag{14}$$

Because $\sum_{(i,j)} w(i, j) = 1$, then $\underline{w}$ is well defined in $H(\mathcal{E})$ (the convergence in (14) is understood with respect to the norm of $H(\mathcal{E})$). On the other hand, the $w(i, j)$'s may be described by (12) as

$$w(i, j) = \sum_{k=1}^{\infty} w_{c_k} J_{c_k}(i, j) , \quad i, j \in S ,$$

where w_{c_k}, $k = 1, 2, \ldots$, are the representative cycle weights (see [4], pp. 34–36). Further define for each $n = 1, 2, \ldots$,

$$\underline{w}(n) = \sum_{k=1}^{n} w_{c_k} \underline{c}_k . \tag{15}$$

Then $\underline{w}(n) \in \mathcal{C}$. Because $\Gamma = \{\underline{\gamma}_1, \underline{\gamma}_2, \ldots\}$ is a base of $\mathcal{C}$, $\underline{w}(n)$ may be written as a linear expression of the form

$$\underline{w}(n) = \sum_{l=1}^{m(n)} w_{\gamma_{k(l)}}(n)\underline{\gamma}_{k(l)}\,, \qquad n = 1, 2, \ldots;\ w_{\gamma_{k(l)}}(n) \in R\,, \tag{16}$$

where $m(n)$ is the corresponding length of summation.

Introduce

$$\begin{aligned} \hat{w}_{\gamma_s}(n) &= w_{\gamma_{k(l)}}(n), && \text{if } s = k(l), \text{ for some } l = 1, 2, \ldots, m(n)\,, \\ &= 0, && \text{otherwise.} \end{aligned}$$

Then

$$\underline{w}(n) = \sum_{s=1}^{m(n)} \hat{w}_{\gamma_s}(n)\underline{\gamma}_s\,, \qquad \hat{w}_{\gamma_s}(n) \in R,\ s = 1, \ldots, m(n)\,, \tag{17}$$

and

$$\left(|\underline{w}(n)|_2\right)^2 = \sum_{s=1}^{m(n)} \left(\hat{w}_{\gamma_s}(n)\right)^2\,.$$

Because the sequence $\{\underline{w}(n)\}_n$ is a Cauchy sequence in $H(\mathcal{E})$, it converges to a vector

$$\underline{w}' = \sum_{(i,j)} w'(i,j) b_{(i,j)} \in H(\mathcal{E})\,,$$

that is,

$$\left|\underline{w}(n) - \underline{w}'\right|_1 \to 0\,, \qquad \text{as } n \to \infty\,.$$

The formal expression of $\underline{w}(n)$ in $H(\mathcal{E})$ is as follows:

$$\begin{aligned} \underline{w}(n) &= \sum_{(i,j)} \left(\sum_{k=1}^{n} w_{c_k} J_{c_k}(i,j) \right) b_{(i,j)} \\ &= \sum_{(i,j)} \left(\sum_{s=1}^{m(n)} \hat{w}_{\gamma_s}(n) J_{\gamma_s}(i,j) \right) b_{(i,j)}\,. \end{aligned}$$

Because

$$\left(|\underline{w}(n) - \underline{w}'|_1\right)^2 = \sum_{(i,j)} \left(\sum_{k=1}^{n} w_{c_k} J_{c_k}(i,j) - w'(i,j) \right)^2 \to 0\,,$$

as $n \to \infty$, we obtain that

$$w'(i,j) = \sum_{k=1}^{\infty} w_{c_k} J_{c_k}(i,j)\,,$$

for any $i, j \in S$. Therefore, $\underline{w}' = \underline{w}$ and we may write

$$\underline{w} = \lim_{n\to\infty} \sum_{s=1}^{m(n)} \hat{w}_{\gamma_s}(n)\underline{\gamma}_s ,$$

where convergence is understood in $H(\mathcal{E})$. In general, using the orthogonal decomposition (13) for any $\underline{w}$ in $H(\mathcal{E})$, we may find a sequence

$$\left\{ \sum_{k=1}^{m(n)} \tilde{w}_k(n)\underline{\gamma}_k \right\}_n$$

of elements of C and a vector $\underline{u}$, orthogonal on C, such that

$$\underline{w} = \lim_{n\to\infty} \sum_{k=1}^{m(n)} \tilde{w}_k(n)\underline{\gamma}_k + \underline{u} ,$$

where convergence is understood in $H(\mathcal{E})$.

Our results can now be summarized in the following theorem.

Theorem 3. *Given a denumerable set S, let $P = (p_{ij}, i, j \in S)$ be an irreducible positive-recurrent stochastic matrix that satisfies the local-finiteness condition, and let $\{C, w_c\}$ be a cycle representation of P with respect to the invariant probability distribution $\pi = \{\pi_i, i \in S\}$. Let $C = \bigcup_{n=1}^{\infty} C^n$, $C^n = \{c_1, c_2, \ldots, c_n\}$, $n = 1, 2, \ldots$.*

(i) *Then P and $\{C, w_c\}$ define, respectively, a Fourier series*

$$\underline{w} = \sum_{(i,j)} (\pi_i p_{ij}) b_{(i,j)}$$

on the edges $b_{(i,j)}$ of the graph of P, and a sequence

$$\underline{w}(n) = \sum_{s=1}^{m(n)} \hat{w}_{\gamma_s}(n)\underline{\gamma}_s , \quad n = 1, 2, \ldots,$$

of Fourier representations on the independent cycles $\{\underline{\gamma}_s\} = \Gamma$ describing C^n, $n = 1, 2, \ldots$, where each $\hat{w}_{\gamma_s}(n)$ is a linear expression of the cycle weights w_c, $c \in C^n$.

(ii) *We have*

$$\underline{w} = \lim_{n\to\infty} \sum_{s=1}^{m(n)} \hat{w}_{\gamma_s}(n)\underline{\gamma}_s , \quad in\ H(\mathcal{E}) .$$

In general, for any $\underline{w}$ in $H(\mathcal{E})$ there exist a sequence $\left\{\sum_{k=1}^{m(n)} \tilde{w}_k(n)\underline{\gamma}_k\right\}_n$ of Fourier representations on the cycles of Γ, and a vector $\underline{u}$, orthogonal on C, such that

$$\underline{w} = \lim_{n\to\infty} \sum_{k=1}^{m(n)} \tilde{w}_k(n)\underline{\gamma}_k + \underline{u} .$$

References

[1] P. J. Hilton and S. Wylie, *Homology Theory*, Cambridge University Press, Cambridge, U.K., 1967.

[2] S. Kalpazidou, Asymptotic behaviour of sample weighted circuits representing recurrent Markov chains, *J. Appl. Probability* **9** (1981), 545–556.

[3] S. Kalpazidou, On the rotational dimension of stochastic matrices, *Ann. Probability* **23**(2) (1995), 966–975.

[4] S. Kalpazidou, *Cycle Representations of Markov Processes*, Springer-Verlag, New York, 1995.

[5] S. Lefschetz, Applications of algebraic topology, *Applied Mathematical Sciences*, no. 16, Springer-Verlag, New York, 1975.

[6] Qian Minping and Qian Min, Circulation for recurrent chain, *Z. Wahrsch. Verw. Gebiete* **58** (1982), 203–210.

[7] F. Riesz and Sz. B. Nagy, *Leçons d'Analyse Fonctionelle*, Académiai Kiadó, Budapest, Hungary, 1952.

[8] A. H. Zemanian, *Distribution Theory and Transform Analysis*, Dover, New York, 1965.