MATHEMATICAL DEMOGRAPHY: RECENT DEVELOPMENTS IN POPULATION PROJECTIONS

Joel E. COHEN
Rockefeller University
New York, U.S.A.

Mathematical demography has flowered in many different gardens during the past twenty years. Several reviews of mathematical demography are available: Keyfitz (1968), Feiçhtinger (1971), Pollard (1973), Ludwig (1978), Keiding (1975), Menken (1977), Smith and Keyfitz (1977), Kurtz (1981), and Cohen (1984). This necessarily brief review deals only with stochastic models for population projection developed since 1965.

POPULATION PROJECTION MODELS

Stochastic models are needed for population projection because deterministic models fail to account for the variability of historical demographic data and to provide probabilistically meaningful estimates of the uncertainty of demographic predictions. Stochastic population projection models may include migration, mortality and fertility (Lee, 1978b; McDonald, 1981).

Among stochastic projection models, one can distinguish, not always sharply, between structural models and time-series models. Structural models represent some underlying mechanism of population growth. Time-series models apply to demographic data general techniques in which the form of the model need not be based on demographic theory.

Structural projection models usually describe either or both of two sources of random fluctuation: demographic variation and environmental variation. Demographic variation arises from the stochastic operation of mechanisms with fixed vital rates. Environmental variation arises when the demographic rates themselves are governed by a stochastic process.

Population Projection with Demographic Variation

Deterministic projections (Siegel, 1972) of populations closed to migration commonly use the recurrence relation

$$y(t+1) = L(t+1)\, y(t), \quad t = 0, 1, 2, \ldots$$

where $y(t)$ is a vector in which the ith component is the number of females in age class i at time t, $i = 1, \ldots, k$; $y(0)$ is a given initial age census of the female population; and $L(t)$ is a $k \times k$ non-negative matrix, conventionally called a Leslie matrix. See Keyfitz (1968) or Pollard (1973) for details.

Taking L to be independent of t, i.e. constant in time, Pollard (1966) reinterprets (1) as a multitype branching process. The survival and fertility of each female are assumed independent of each other and of the survival and fertility of all other females. Then $y(t)$ in (1) can be interpreted as the expectation of the age census at time t. A linear recurrence relation that uses the direct or Kronecker product of two matrices describes the variances and covariances of each census of females. Goodman (1967) computes the probability in Pollard's model that the line of descendants of an individual of any given age will eventually become extinct.

Projections of the Norwegian population as a multitype branching process give estimates of uncertainty that Schweder (1971) considers unrealistically low.

Independently of Pollard (1966), Staroverov (1976) considers exactly the same model. Because the model variances are implausibly small compared to the historical variation in Soviet birth rates, Staroverov replaces the assumption that each individual evolves independently with the assumption that groups of c individuals evolve as units, independently of other groups. As c increases, the variance of numbers in each age group increases while the means remain unaltered. A comparison of observed and projected births from 1960 to 1973 suggests that even $c = 100{,}000$ is too small, and that it is necessary to allow for temporal variation in the fertility and mortality parameters.

With a different interpretation of $L(t)$ in (1) from the usual Leslie matrix, Goodman (1968) describes two sexes; Breev and Staroverov (1977) describe labor force migration; Wu and Botkin (1980) describe elephants. Deistler and Feichtinger (1974) show that the multitype branching process model may be viewed as a special case of a model of additive errors proposed for population dynamics by Sykes (1969).

Mode (1976) develops population projection models using renewal theory rather than matrix methods. The continuous-time stochastic theory analogous to what has just been described is presented by Keiding and Hoem (1976) and Braun (1978), with extensions to parity-dependent birth rates and multiregional populations. Cohen (1984) discusses the merits of branching processes as models of human and nonhuman populations.

Population Projection with Environmental Variation

In a large population, the effects of demographic variation are normally negligible compared to those of apparent changes in vital rates. Sykes (1969) supposes that, given $L(t+1)$, $y(t)$ determines $y(t+1)$ exactly, but that there is no correlation between $L(t)$ and $L(s), s \neq t$. Sykes computes the means and covariances of the age censuses, allowing the means and covariances of the sequence $\{(L(t)\}$ to be inhomogeneous in time. Seneta (1972) pursues the computation of variances in the models of Sykes (1969) and Pollard (1966). Lee (1974) discusses the numerical example that Sykes gives.

LeBras (1971, 1974) and Cohen (1976, 1977) develop demographic applications of products of random matrices (Furstenberg and Kesten, 1960). Under exactly stated conditions, Furstenberg and Kesten prove theorems that imply that the census vector $y(t)$ changes asymptotically exponentially and that the elements of $y(t)$, namely the numbers of females in each age class, are, for large t, asymptotically lognormal.

Cohen (1977) gives conditions under which the probability distribution of age structure asymptotically becomes independent of initial conditions. This weak stochastic ergodic theorem is the probabilistic analog of the deterministic weak ergodic theorem of Coale and Lopez (Pollard, 1973, pp. 51-55).

The theory of products of random matrices as models of environmental variability in age-structured populations is developed by, among others, Cohen (1980), Lange (1979), Tuljapurkar and Orzack (1980), and Tuljapurkar (1984).

An elementary but important observation emerging from these studies is a distinction between two measures of the long-run rate of growth of a population in a stochastic environment. One measure, studied by Furstenberg and Kesten (1960), is the average of the long-run rates of growth along each sample path, $\log \lambda = \lim_{t \uparrow \infty} t^{-1} E(\log y(t))$. Another measure is the long-run rate of growth of the average population, $\log \mu = \lim_{t \uparrow \infty} t^{-1} \log E(y_1 \ldots (t))$.

For deterministic models, $\lambda = \mu$, but in general, in stochastic models, $\lambda \leq \mu$ with strict inequality in most examples.

Products of random matrices provide a natural model of age-structured populations of the game fish, striped bass (Cohen et al., 1983), and can be used to establish a long-run decline in populations of the striped bass samples in the Chesapeake Bay (Goodyear et al., 1984). Heyde and Cohen (1984) use martingale limit theorems to estimate confidence intervals for demographic projections based on products of random matrices assuming only ergodicity and stationarity in the environment, and apply their methods to the striped bass.

When the models based on products of random matrices are applied to human data, an updating of the series of Swedish population sizes analyzed by Saboia (1974), the confidence intervals estimated according to the techniques of Heyde and Cohen (1984) are broader, i.e., unnecessarily pessimistic, compared to confidence intervals estimated empirically (Cohen, submitted).

Population Projection with Demographic and Environmental Variation

Demographic variation and environmental variation can be modeled together. If the probabilities of giving birth and of surviving in a multitype branching process are themselves random variables (Pollard, 1968; Bartholomew, 1975), the moments of the number of individuals in each age class can be computed from a modification of a recurrence relation derived by Pollard (1966).

For a multitype branching process such that the offspring probability generating functions at all times are independently and identically distributed, Weissner (1971) gives some necessary and some sufficient conditions for almost sure extinction of the population (see also Namkoong, 1972; Athreya and Karlin, 1971). Weissner, Athreya and Karlin do not discuss the application of these results to age-structured populations.

Time-Series Models

The application of modern stochastic time-series methods to demographic data originates with Lee (1970) and Pollard (1970). Lee (1978a, for summary) uses long time series, for example, of births and marriages or of mortality and wages, to test alternative historical theories of demographic and economic dynamics. Pollard (1970) develop a second-order autoregressive model of the growth rate of total population size for Australia.

Lee's (1974) analysis of births from 1917 to 1972 in the United States demonstrates that the distinction between structural models and time-series models is not sharp. Eq. (1) implies that each birth may be attributed to the fertility of the survivors of some preceding birth cohort. Hence the sequence of births $\{y_1(t)\}$ is described by a renewal equation. By a sequence of approximations to this renewal equation, Lee transforms the residuals of births from their long-run trend into an autoregressive process for which variations in the net reproduction rate are the error term.

Independently of Lee, Saboia (1974) develops autoregressive moving average (ARMA) models using Box-Jenkins techniques for the total population of Sweden. Based on data from 1780 to 1960 at 5-year intervals, his projections for 1965 compare favorably with some standard demographic projections.

Saboia (1977) relates ARMA models to the renewal equation for forecasting births. In these models, the age-specific vital rates can vary over time; migration is recognized. Using female birth time-series for Norway, 1919-1975, he gives forecasts with confidence intervals up to 2000. However, Saboia's (1977) models are not the simplest required to describe the data (McDonald, 1980).

McDonald (1979) describes the relationships among the renewal equation model, with migration added, structural stochastic econometric models, and ARMA models. He suggests that exogenous, perhaps economic, variables will have to be invoked to explain a sharp decline that occurred in the number of Australian births after 1971. Land (1980) similarly suggests incorporating exogenous variables in structural stochastic projection models with environmental variation.

The forecasts of the time-series models have very wide confidence intervals (e.g., McDonald, 1979; McNeil, 1974). In view of the uncertainty of the demographic future, policy that depends on population size and structure should be flexible enough to allow for different possible futures.

In addition to spectral methods and Box-Jenkins techniques, other recent approaches to population time-series modelling include a stochastic version of the logistic equation (McNeil, 1974) as a model of United States Census total population counts; the Karhunen-Loève procedure (Basilevsky and Hum, 1979) for quarterly records of births in two Jamaican parishes, 1880 to 1938; and an age- and density-dependent structural model, estimated by use of the Kalman-Bucy filter (Brillinger et al., 1980), for age-aggregated counts of the sheep blow-fly.

ASSESSMENT AND PROSPECTS

Hajnal (1957) raises profound doubts about the possibility of projecting the future of populations (also see Hoem, 1973). There is even a joke, not told by demographers (Heaven forbid!), but by consumers of demographic projections.

Question: What is the difference between a demographer and a mathematical demographer ?

Answer: A demographer is somebody who guesses wrong about the future of populations. A mathematical demographer is somebody who uses mathematics and computers to guess wrong about the future of populations.

Greater effort needs to be made to evaluate quantitatively the merits and demerits of population projection techniques and the underlying models on which they rest, following the path of Henry and Gutierrez (1977), Ascher (1978), Keyfitz (1982), Stoto (1983), Stoto and Schrier (1982), and Smith (1984). If these efforts succeed, demographers can replace the old joke with a new one.

Question: What is the difference between a bad mathematical demographer and a good one ?

Answer: A bad mathematical demographer uses mathematics and computers to guess wrong about the future of populations. A good mathematical demographer uses mathematics and computers to guess wrong about the future of populations, but tells you reliably how far wrong you can expect him to be.

SUMMARY

Mathematical demography has flowered recently in more areas than can be reviewed here. This review deals only with stochastic models for population projection developed since 1965.

Stochastic models are needed for population projection because deterministic models fail to account for the variability of historical demographic data. Deterministic models also fail to provide probabilistically meaningful estimates of the uncertainty of demographic predictions.

Among stochastic projection models, one can distinguish, not always sharply, between structural models and time-series models. Structural models represent some supposed underlying mechanism of population growth. Time-series models apply to demographic data general techniques in which the form of the model need not be based on demographic theory.

Structural projection models usually describe either or both of two sources of random fluctuation: demographic variation and environmental variation. Demographic variation arises from the stochastic operation of mechanisms with fixed vital rates. Environmental variation arises when the demographic rates themselves are governed by a stochastic process.

Deterministic projections of populations closed to migration commonly use the recurrence relation $y(t+1) = L(t+1)y(t)$, $t = 0, 1, 2, \ldots$, where $y(t)$ is a vector in which the ith component is the number of females in age class i at time t, $i = 1, \ldots, k$; $y(0)$ is a given initial age census of the female population; and $L(t)$ is a $k \times k$ non-negative matrix, conventionally called a Leslie matrix. To model demographic variation, this recurrence relation has been interpreted in terms of multitype branching processes.

However, the effects of demographic variation are normally negligible compared to those of apparent changes in vital rates. To model environmental variation, the projection matrix L(t) may be considered to be random. Under reasonable conditions, the census vector y(t) changes asymptotically exponentially and the elements of y(t), namely, the numbers of females in each age class, are, for large t, asymptotically lognormal. Moreover, the probability distribution of age structure asymptotically becomes independent of initial conditions. This weak stochastic ergodic theorem is the probabilistic analog of the deterministic weak ergodic theorem of Coale and Lopez. Statistical methods for models with environmental variability exist.

The application of modern stochastic time-series methods to demographic data began in 1970. Some projections based on time-series methods compare favorably with some standard demographic projections. The forecasts of the time-series models have very wide confidence intervals. In view of the uncertainty of the demographic future, policy that depends on demographic forecasts should be flexible enough to allow for different possible futures.

Greater effort needs to be made to evaluate quantitatively the merits and demerits of population projection techniques and the underlying models on which they rest.

RESUME

DEMOGRAPHIE MATHEMATIQUE: APPLICATIONS RECENTES DANS LE DOMAINE DES PROJECTIONS DEMOGRAPHIQUES

La démographie mathématique s'est développée récemment dans tellement de domaines qu'il est impossible de les passer en revue, seuls les modèles stochastiques pour les projections démographiques élaborées depuis 1965 seront examinées ici.

Il est nécessaire d'avoir recours aux modèles stochastiques pour l'élaboration des projections démographiques parce que les modèles déterministes ne parviennent pas à rendre compte de la variabilité des données démographiques dans le temps. Les modèles déterministes ne parviennent pas non plus à fournir des estimations de l'incertitude des prédictions démographiques qui ont un sens du point de vue probabiliste.

Parmi ces modèles stochastiques, il est possible de distinguer, quoique pas toujours de manière nette, les modèles structurels des modèles basés sur des séries chronologiques. Les modèles structurels représentent l'un ou l'autre mécanisme qui est supposé régir la croissance démographique. Les modèles basés sur des séries chronologiques appliquent aux données démographiques des techniques générales qui n'exigent pas que la forme du modèle soit basée sur la théorie démographique.

Les modèles structurels de projection décrivent généralement sinon les deux sources de variation aléatoire du moins l'une d'entre elles, à savoir: la variation démographique et la variation de l'environnement. La variation

démographique résulte du jeu de l'action stochastique des mécanismes associés à des taux démographiques fixés au préalable. La variation de l'environnement intervient lorsque les taux démographiques eux-mêmes sont gouvernés par un processus stochastique.

Les projections déterministes de populations fermées utilisent communément la relation de récurrence $y(t+1) = L(t+1)y(t)$, $t = 0, 1, 2, \ldots$, où $y(t)$ est un vecteur sur lequel la 1ère composante est le nombre de femmes appartenant à la classe d'âge i au temps t , $i = 1, \ldots, k$; $y(o)$ est la distribution par âge initiale de la population féminine obtenue lors d'un recensement donné; et $L(t)$ est une matrice $k \times k$ non-négative, appelée par convention la matrice de Leslie. Pour rendre la variation démographique sous forme de modèles, il faut interpréter la relation de récurrence en termes de processus à dérivations multiples.

Normalement, les effets de la variation démographique restent cependant négligeables comparativement à ceux produits par les changements apparents des taux démographiques. Pour rendre compte de la variation de l'environnement sous forme de modèles, la matrice de projection $L(t)$ sera considérée comme aléatoire. Sous certaines conditions raisonnables, le vecteur du recensement $y(t)$ change (de manière asymptotique et exponentielle) et les éléments de $y(t)$, à savoir les nombres de femmes appartenant à chaque classe d'âges, sont asymptotiquement lognormaux pour une valeur de t .

De plus, la distribution probabiliste de la structure par âges devient asymptotiquement indépendante des conditions initiales. Ce théorème de faible ergodicité stochastique est le correspondant probabiliste du théorème de faible ergodicité déterministe. Enoncé par Coale et Lopez, il existe des méthodes statistiques qui s'appliquent aux modèles incorporant la variabilité de l'environnement.

C'est en 1970 que l'application aux données démographiques, des méthodes stochastiques modernes basées sur des séries chronologiques a commencé. Certaines projections calculées selon des méthodes basées sur des séries chronologiques se comparent avantageusement à certaines projections démographiques courantes. Les prévisions de ces modèles basés sur des séries chronologiques ont de très grands intervalles de confiance. Etant donné l'incertitude de l'avenir démographique, toute politique qui se fonde sur des prévisions démographiques devrait être suffisamment flexible pour s'adapter à différents avenirs possible.

Une attention plus grande devrait être accordée à l'évaluation quantitative du pour et du contre des différentes techniques démographiques ainsi que des modèles qui les sous-tendent.

REFERENCES

Ascher, W., 1978. Forecasting: An Appraisal for Policy-Makers and Planners, Johns Hopkins University Press, Baltimore.

Athreya, K. and Karlin, S., 1971. On branching processes with random environments: I. Extinction probabilities. Ann. Math. Stat., Vol. 42, pp. 1499-1520.

Bartholomew, D.J., 1975. Errors of prediction for Markov chain models, J. Roy. Stat. Soc. B37, pp. 444-456.

Basilevsky, A. and Hum, D.P.J., 1979. Karhunen-Loève analysis of historical time series with an application to plantation births in Jamaica. J. Amer. Stat. Assoc., Vol. 74, pp. 284-290.

Braun, H.I., 1978. Stochastic stable population theory in continuous time, Scand. Actuarial J., pp. 185-203.

Breev, B.D. and Staroverov, O.V., 1977. O metode ucheta faktorov pri prognoze dvizheniia naseleniia i trudovykh resursov. Ekonomika i matematicheskie metody 13, pp. 489-499. English translation: Determinants in forecasting the movement of population and labor resources, Matekon 14, pp.80-97, Spring, 1978.

Brillinger, D.R., Guckenheimer, J., Guttorp, P., and Oster, G., 1980. Empirical modelling of population time series data: the case of age and density dependent vital rates. Lect. Math. Life Sci., Vol. 13, pp. 65-90, American Mathematical Society, Providence.

Cohen, Joel E., 1976. Ergodicity of age structure in populations with Markovian vital rates, I: Countable states, J. Amer. Stat. Assoc., Vol. 71, pp. 335-339.

Cohen, Joel E., 1977. Ergodicity of age structure in populations with Markovian vital rates, II: General states, III: Finite-state moments and growth rates; illustration, Adv. Appl. Probab., Vol. 9, pp. 18-37; pp. 462-475.

Cohen, Joel E., 1980. Convexity properties of products of random nonnegative matrices, Proc. Nat. Acad. Sci. U.S.A., Vol. 77, pp. 3749-3752.

Cohen, J.E., Christensen, S.W. and Goodyear, C.P., 1983. A stochastic age-structured population model of striped bass (*Morone saxatilis*) in the Potomac River. Canadian J. Fisheries and Aquatic Sciences, Vol. 40, pp. 2170-2183.

Cohen, J.E., 1984. Stochastic demography. In: Encyclopedia of the Statistical Sciences, S. Kotz and N.L. Johnson, eds., John Wiley, New York.

Cohen, J.E., submitted, Swedish population forecasts and confidence intervals: products of random matrices and an empirical approach.

Deistler, Manfred and Feichtinger, Gustav, 1974. The linear model formulation of a multitype branching process applied to population dynamics, J. Amer. Stat. Assoc., Vol. 69, pp. 662-664.

Feichtinger, G., 1971. Stochastische Modelle demographischer Prozesse, (Stochastic Models of Demographic Processes), Springer, Berlin.

Furstenberg, H. and Kesten, H., 1960. Products of random matrices, Ann. Math. Stat., Vol. 31, pp. 457-469.

Goodman, L.A., 1967. The probabilities of extinction for birth-and-death processes that are age-dependent or phase-dependent, Biometrika, Vol. 54, pp. 579-596.

Goodman, L.A., 1968. Stochastic models for the population growth of the sexes, Biometrika, Vol. 55, pp. 469-487.

Goodyear, C.P., Cohen, J.E. and Christensen, S.W., 1984. Maryland striped bass: recruitment declining below replacement, Trans Amer. Fish. Soc., November.

Hajnal, J., 1957. Mathematical models in demography, Cold Spring Harbor Symposia on Quantitative Biology 22, pp. 97-103.

Henry, Louis and Gutierrez, Hector, 1977. Qualité des prévisions démographiques à court terme. Etude de l'extrapolation de la population totale des départements et villes de France, 1821-1975. (Quality of short-term demographic forecasts. Study of the extrapolation of the total population of the departments and cities in France, 1821-1975), Population, Vol. 32, pp. 625-647, Paris.

Heyde, C.C. and Cohen, J.E., 1984. Confidence intervals for demographic projections based on products of random matrices, Theor. Pop. Biol., Dec.

Hoem, J.M., 1973. Levels of error in population forecasts, Statistisk Sentralbyra, Art. 61, Oslo.

Keiding, N., 1975. Extinction and exponential growth in random environments, Theor. Pop. Biol., Vol. 8, pp. 49-63.

Keiding, N. and Hoem, J., 1976. Stochastic stable population theory with continuous time, I. Scand. Actuarial J., pp. 150-175.

Keyfitz, N., 1968. Introduction to the Mathematics of Population, Addison-Wesley, Reading.

Keyfitz, N., 1982. The limits of population forecasting. Chapter 13 in Keyfitz N. 1982, Population Change and Social Policy, Abt Books, Cambridge, MA.

Kurtz, T.G., 1981. Approximation of Population Processes, Society for Industrial and Applied Mathematics, Philadelphia.

Land, Kenneth C., 1980. Modeling macro social change. In: Karl F. Schuessler, ed., Sociological Methodology, Jossey-Bass, San Francisco, pp. 219-278.

Lange, K., 1979. On Cohen's stochastic generalization of the strong ergodic theorem of demography. J. Appl. Probab., Vol. 16, pp. 496-504.

LeBras, H., 1971. Eléments pour une théorie des populations instables, (Elements of a theory of unstable populations), Population, Vol. 26, pp. 525-572, Paris.

LeBras, H., 1974. Populations stables aléatoires, (Stochastic stable populations), Population, Vol. 29, pp. 435-464, Paris.

Lee, R.D., 1970. Econometric studies of topics in demographic history, Unpublished Ph.D. dissertation, Harvard University, Cambridge.

Lee, R.D., 1974. Forecasting births in post-transition populations: stochastic renewal with serially correlated fertility, J. Amer. Stat. Assoc., Vol. 69, pp. 607-617.

Lee, R.D., 1978a. Models of preindustrial population dynamics with application to England. In: Charles Tilly, ed. , Historical Studies of Changing Fertility, Princeton Univ. Press, pp. 155-207, Princeton.

Lee, R.D., 1978b. New methods for forecasting fertility: an overview, Population Bull. U.N. 11, pp. 6-11.

Ludwig, D., 1978. Stochastic Population Theories, Springer, New York.

McDonald, J., 1979. A time series approach to forecasting Australian total live-births, Demography, Vol. 16, pp. 575-601.

McDonald, J., 1980. Births time series models and structural interpretations, J. Amer. Stat. Assoc., Vol. 75, pp. 39-41.

McDonald, J., 1981. Putting fertility explanations in perspective, Project Forecast: Dynamic Modelling, Flinders U., S. Australia, Res. Rep. 22.

McNeil, D.R., 1974. Pearl-Reed type stochastic models for population growth, Theor. Pop. Biol., Vol. 5, pp. 358-365.

Menken, J.A., 1977. Current status of demographic models, Population Bull. U.N. 9, pp. 22-34.

Mode, C.J., 1976. Age-dependent branching processes and sampling frameworks for mortality and marital variables in nonstable populations, Math. Biosciences, Vol. 30, pp. 47-67.

Namkoong, G., 1972. Persistence of variances for stochastic, discrete-time population growth models, Theor. Pop. Biol., Vol. 3, pp. 507-518.

Pollard, J.H., 1966. On the use of the direct matrix product in analyzing certain stochastic population models, Biometrika, Vol. 53, pp. 397-415.

Pollard, J.H., 1968. A note on multi-type Galton-Watson processes with random branching probabilities, Biometrika, Vol. 55, pp. 589-590.

Pollard, J.H., 1970. On simple approximate calculations appropriate to populations with random growth rates, Theor. Pop. Biol., Vol. 1, pp. 208-218.

Pollard, J.H., 1973. Mathematical Models for the Growth of Human Populations, University Press, Cambridge.

Sabola, J., 1974. Modeling and forecasting populations by time-series: the Swedish case, Demography, Vol. 11, pp. 483-492.

Sabola, J., 1977. Autoregressive integrated moving average (ARIMA) models for birth forecasting, J. Amer. Stat. Assoc., Vol. 72, pp. 264-270.

Schweder, T., 1971. The precision of population projections studied by multiple prediction methods, Demography, Vol. 8, pp. 441-450.

Seneta, E., 1972. Population projection variances and path analysis, J. Amer. Stat. Assoc., Vol. 67, pp. 617-619.

Siegel, J.S., 1972. Development and accuracy of projections of population and households in the United States, Demography, Vol. 9, pp. 51-68.

Smith, D. and Keyfitz, N., 1977. Mathematical Demography: Selected Papers, Springer, New York.

Smith, S.K., 1984. Confidence intervals for county population forecasts, Paper presented at annual meeting of Population Assoc. America, May.

Staroverov, O.V., 1976. Interval'nyi prognoz struktury naseleniia. Ekonomika i matematicheskie metody 12, pp. 56-71. English translation: Interval estimates of population structure, Matekon, Vol. 13, pp. 42-67, Winter 1976-77.

Stoto, M.A., 1983. The accuracy of population projections, Journal of the American Statistical Association, Vol. 78, pp. 13-20.

Stoto, M.A. and Schrier, A.P., 1982. The accuracy of state population projections, John F. Kennedy School of Government, Discussion Paper 117D, Harvard University, Cambridge, Mass.

Sykes, Z.M., 1969. Some stochastic versions of the matrix model for population dynamics, J. Amer. Stat. Assoc., Vol. 64, pp. 111-130.

Tuljapurkar, S.D. and Orzack, S.H., 1980. Population dynamics in variable environments, I. Long-run growth rates and extinction, Theor. Pop. Biol., Vol. 18, pp. 314-342.

Tuljapurkar, S.D., 1984. Demography in stochastic environments, I. Exact distributions of age structure, J. Math. Biology, Vol. 19, pp. 335-350.

Weissner, E.W., 1971. Multitype branching processes in random environments, J. Appl. Probab., Vol. 8, pp. 17-31.

Wu, L.S.-Y. and Botkin, D.B., 1980. Of elephants and men: a discrete, stochastic model for long-lived species with complex life histories, Amer. Nat., Vol. 116, pp. 831-849.