

## When Will a Large Complex System Be Stable?

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May (1972, 1973) and Hastings (1982a, b, 1983a, b) announced criteria for the probable stability or instability, as  $n \uparrow \infty$ , of systems of  $n$  linear ordinary differential equations or difference equations with random coefficients fixed in time. However, simple, explicit counter-examples show that, without some additional conditions, the claims of May and Hastings can be false.

May (1972, 1973) announced criteria for the probable stability or instability of a system of  $n$  linear ordinary differential equations with random coefficients fixed in time, as  $n$  increases to infinity. He gave these criteria the ecological interpretation that, as the number  $n$  of species in an ecological community increases, increasingly severe constraints must be imposed on the distribution of the interspecific interaction coefficients to assure the stability of the community. The constraints could be satisfied, as  $n$  increases, by reducing either the connectance (the fraction of interaction coefficients that are not zero) or the variance of the non-zero interaction coefficients. This interpretation assumed that the interaction coefficients between species can usefully be described as random variables fixed in time. May's claims attracted attention because they appear to contradict an ecological folk law that increased complexity (measured by the number  $n$  of interacting species or the connectance among them) promotes increased stability. Pimm (1984) has comprehensively reviewed the vexed question of the complexity and stability of ecological systems.

Recently Hastings (1982a, b, 1983a, b) announced variants and proofs of May's claims.

The purpose of this note is to point out that some simple, explicit counter-examples (Cohen & Newman 1984a) show that May's and Hastings's claims are false in the generality with which they are stated. The counter-examples are quite robust and include broad sets of models that

fall within the hypotheses of May and Hastings. In the models we have constructed, May's and Hastings's announced criteria for stability and other assumptions are satisfied but the probability that the system of differential or difference equations is stable does not approach 1 as  $n$  increases. In another example, based on Hastings's use of, difference equations, the announced criteria for instability and other assumptions are satisfied, but the probability that the system is unstable approaches zero as  $n$  increases.

By way of illustration, we shall focus here on one specific claim made by Hastings (1982a) and the simplest of our counter-examples.

Hastings (1982a) considered the time-homogeneous system of difference equations

$$x_{t+1} = Bx_t \quad (1)$$

where  $x_t$  is a real  $n$ -vector and the random  $n \times n$  matrix  $B$  has all elements independently and identically distributed and fixed in time. He defined the system (1) to be stable if the spectral radius  $r(B)$ , the largest of the absolute values of the eigenvalues of  $B$ , is less than 1.

Hastings supposed that each element  $B_{ij}$  of  $B$  is 0 with probability  $1 - C$ ,  $0 \leq C \leq 1$ , and with probability  $C$  is drawn, once and for all, from a distribution with mean and all odd moments 0 and variance  $a^2$ . The diagonal elements of  $B$  have the same distribution as the off-diagonal elements of  $B$ . In the course of his argument, Hastings assumed that, as  $n$  increases,  $a$  is fixed and  $C = k/n$ , where  $k$  is a fixed positive constant. He suggested that data justify the assumption that  $C = k/n$  in the context where  $n$  is the number of species in an ecological community and  $B$  is a matrix that describes the species' interactions, which are assumed to be fixed in time. Under these assumptions, he claimed that if  $nCa^2 = n \text{Var } B_{ij} < 1$  then  $r(B) < 1$  with probability approaching 1 as  $n \uparrow \infty$ , while if  $nCa^2 > 1$  then  $r(B) > 1$  with probability approaching 1 as  $n \uparrow \infty$ .

For a counter-example to Hastings's claims concerning the stability of the system (1), choose any finite positive constant  $k$ . Suppose  $X_{ij}$  are independent normally distributed real random variables with mean 0 and positive variance  $a^2$ . Then the probability  $p$  that any one of the  $X_{ij}$ 's exceeds 1 in absolute value is positive. Let  $C_n = k/n$  and let each element  $B_{ij}$  of the matrix  $B$  be independently and identically distributed, equal to 0 with probability  $1 - C_n$  and equal to  $X_{ij}$  with probability  $C_n$ . We have proved that the probability of instability does not approach 0 as  $n$  increases, even though we can choose  $ka^2 = nC_n a^2 < 1$ , contrary to Hastings (1982a).

The assumption in this counter-example that each  $X_{ij}$  is normally distributed may be replaced by the much more general assumption that each  $X_{ij}$  has a positive probability of exceeding 1 in absolute value.

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H. M. Hastings has suggested (personal communication, January 1984) that one can understand such counter-examples to asymptotic stability in terms of the connectedness properties of the random (directed) graph obtained from the matrix  $B$  by placing a directed edge from  $i$  to  $j$  if  $B_{ij} \neq 0$ .

In the spirit of that suggestion, we note that counter-examples to asymptotic instability also follow from certain connectedness properties. When each edge occurs with probability  $k/n$  and  $k < 1$ , it is known (see Cohen & Newman 1984b) that there is, as  $n \rightarrow \infty$ , a strictly positive probability that the graph contains *no* (directed) cycles. It is easy to see that the corresponding matrix  $B$  then has  $B^m = 0$  for a sufficiently large power  $m$  so that  $r(B) = 0$ . It follows that the probability of stability does not approach 0 as  $n$  increases, even though we can choose  $ka^2 = nC_n a^2 > 1$ , contrary to Hastings's (1982a) claim concerning asymptotic instability.

Counter-examples to asymptotic instability which do not require  $k < 1$  are also possible. In particular, we have constructed (Cohen & Newman 1984a) a sequence of random matrices  $B$  with independently and identically distributed elements  $B_{ij}$  such that  $\lim_{n \uparrow \infty} n \text{Var } B_{ij} = \infty$  while  $\lim_{n \uparrow \infty} P(r(B) < 1) = 1$ , again contrary to Hastings (1982a).

Though they fall within the hypotheses of May and Hastings, our counter-examples to their instability claims may be ecologically artificial in that we require either that  $k$  be less than 1 or else that  $X_{ij}$  can take on very large values (with small probability).

Our counter-examples leave open the possibility that May's conjectures may be true under certain conditions as suggested by numerical studies (McMurtrie, 1975 and D. E. McClure, personal communication, June 1984).

Recently Geman (1984) has given conditions sufficient to prove the stability part of May's conjectures. Specifically, let  $\{b_{ij}\}$ ,  $i = 1, 2, \dots, j = 1, 2, \dots$ , be independent and identically distributed random variables with mean zero and finite standard deviation  $\sigma$ . For each  $n$ , let  $B_n$  be the  $n \times n$  matrix with  $i, j$  element  $b_{ij}$ . Geman shows under certain hypotheses on the moments of  $b_{ij}$  that as  $n \uparrow \infty$ , the lim sup of the spectral radius  $r(B_n/n^{1/2})$  is less than or equal to  $\sigma$  with probability 1. Robert S. Maier (personal communication, September 1984) has informed us without providing details that he has established the instability part of May's conjectures under certain conditions.

We have also analyzed a model, different from that considered by May and Hastings, that drops the assumption that the random interaction coefficients between species are fixed in time (Cohen & Newman 1984a). For some cases of this model, which is based on products of random matrices (Furstenberg & Kesten, 1960), stability or instability is determined asymptotically by criteria of exactly the form suggested by May. For other

cases of this same model, criteria different in form from those proposed by May determine the stability or instability of the system.

We agree with this observation of May (1971): "In nature we deal *not* with arbitrary complex systems, but rather with ones selected by a long and intricate process...; mathematical theorems tend to deal with general complex systems, which are quite another matter". However, this is only a tendency of mathematics, not a requirement. Our stability results for products of random matrices may eventually be extended from the case of independent matrices to more realistic stochastic models with nonlinear dynamics and memory.

Since early investigations (Gardner & Ashby 1970) of the stability of random systems were motivated by models of the brain, our analysis of models with randomly fixed and with randomly changing coefficients may be of interest in neurobiology as well as in ecology.

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