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THE ASYMPTOTIC PROBABILITY THAT A RANDOM GRAPH IS A UNIT INTERVAL GRAPH, INDIFFERENCE GRAPH, OR PROPER INTERVAL GRAPH

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Let G be a graph on v labelled vertices with E edges, without loops or multiple edges. Let $v \to \infty$ and let $E = E(v)$ be a function of v such that $\lim E(v)/v^{2/3} = c$. The limit of the probability that a random graph is a unit interval graph, indifference graph or proper interval graph is $\exp(-\frac{4}{3}c^3)$.

Let G be a graph with a finite set A of v labelled vertices and E unlabelled, undirected edges. For x and y in A, $x \neq y$, we write $x\,I\,y$ if (x, y) is an edge of G. Loops and multiple edges are excluded.

G is a unit interval graph [4, p. 144] if and only if, for each x in A, there exists an interval $J(x)$ of the real line, closed and of unit length, such that

$$x\,I\,y \quad \text{if and only if } J(x) \cap J(y) \neq \emptyset. \tag{1}$$

Unit interval graphs have applications in psychology, archeology, political science, economics and other fields [4, 5].

A random graph with v labelled vertices and E edges is defined [2] as one in which the E edges are chosen randomly, without replacement, from the $\binom{v}{2}$ possible edges so that all $C_{v,E}$ possible graphs are equiprobable, where

$$C_{v,E} = \binom{\binom{v}{2}}{E}.$$

If $U_{v,E}$ is the number of graphs on v labelled vertices and E edges that are unit interval graphs, then the probability $\mathbf{P}_{v,E}(\text{u.i.g.})$ that a random graph on v labelled vertices and E edges is a unit interval graph is $U_{v,E}/C_{v,E}$.

We shall describe the behavior of $\mathbf{P}_{v,E}(\text{u.i.g.})$, letting $v \to \infty$ and $E = E(v)$ increase as a function of v.

Theorem. *Suppose*

$$\lim_{v\to\infty} E(v)/v^{2/3} = c. \tag{2}$$

Then

$$\lim_{v\to\infty} \mathbf{P}_{v,E(v)}(\text{u.i.g.}) = \exp(-\tfrac{4}{3}c^3). \tag{3}$$

For large v and E, as long as E^3/v^2 is not very large compared to 1,

$$\mathbf{P}_{v,E}(\text{u.i.g.}) \sim \exp\left(-\binom{v}{4}4p^3(1-p)^3\right), \tag{4}$$

where $p = E/\binom{v}{2}$.

Proof. The proof requires some more definitions and theorems. A graph G is an interval graph if and only if, for each x in A, there exists an interval $J(x)$ of the real line such that (1) holds. (This definition does not require $J(x)$ to be closed or of unit length.) Roberts [4, p. 144] proved that G is a unit interval graph if and only if G is an interval graph and does not contain K_{13} as an induced subgraph. (A graph H is an induced subgraph of G if the vertices and edges of H are among the vertices and edges of G and there is an edge between two vertices of H whenever there is an edge between those two vertices in G.)

The degree $d(G)$ of a graph G with v vertices and E edges is $2E/v$. A graph G is strongly balanced if any proper subgraph of G has a smaller degree than $d(G)$. G is balanced if any subgraph has a degree not greater than $d(G)$.

We see that $d(K_{13}) = \frac{3}{2}$ and that K_{13} is strongly balanced.

Let Z be some property of a graph, and let $\mathbf{P}_{v,E}(Z)$ be the probability that a random graph on v labelled vertices and E edges has the property Z. Let f be a real-valued function of a real argument. Define f to be a threshold function for the property Z if and only if, for any $\varepsilon > 0$ there are positive numbers δ, Δ and v_0 such that for $v > v_0$ and $E \leq \delta f(v)$ we have $\mathbf{P}_{v,E}(Z) < \varepsilon$, while for $v > v_0$ and $E \geq \Delta f(v)$ we have $\mathbf{P}_{v,E}(Z) > 1-\varepsilon$. By a theorem of Erdös and Rényi [2, p. 23], quoted as Theorem A in [1], $v^{2/3}$ is the threshold function for the property that a graph has a subgraph isomorphic to K_{13}. In other words, as $v \to \infty$, the probability that a random graph on v labelled vertices contains at least one subgraph isomorphic to K_{13} approaches 0 if $\lim E(v)/v^b = c$ and $b < 2 - 2/d(K_{13}) = \frac{2}{3}$, while the same probability approaches 1 if $\lim E(v)/v^b = c$ and $b > \frac{2}{3}$. Any balanced subgraph of degree $d > \frac{3}{2}$ has a threshold function $v^{2-2/d} > v^{2/3}$ and hence will appear with asymptotic probability 0 in the limit (2). So any subgraph isomorphic to K_{13} is an induced subgraph with probability 1 in the limit (2), because any additional edges on the given 4 vertices would yield at least one triangle on 3 of the vertices, and a triangle is of higher degree than $\frac{3}{2}$.

Lekkerkerker and Boland [3] proved that a graph G is an interval graph if and only if G contains no induced subgraphs isomorphic to any member of five classes [1, Fig. 1] of forbidden subgraphs. The graphs in all five of these classes are strongly balanced, and therefore balanced, and of degree $d = \frac{12}{7}$ or larger. The threshold functions for the appearance of these forbidden subgraphs in random graphs on v labelled vertices are of the form v^b, where $b \geq 2-2/d = \frac{5}{6}$. Hence these forbidden subgraphs appear with probability 0 in the limit (2). Therefore, in the limit (2), $\mathbf{P}_{v,E(v)}$(u.i.g.) is the probability that a large random graph contains no K_{13} subgraphs. We now compute this probability, using [1, Theorem 2].

Let B be an arbitrary finite class of strongly balanced graphs $G_1, \ldots, G_m$ all having the same degree d. Let the number of (labelled) vertices of G_i be v_i and the number of edges be E_i. Let B_i denote the number of graphs with v_i labelled vertices which are isomorphic to G_i.

Theorem 2. *Let A_k denote the event that a random graph contains exactly k subgraphs each isomorphic to some element of B. Assume that, as the number v of labelled vertices of a random graph is increased, the number $E(v)$ of edges is also increased so that $\lim_{v\to\infty} E(v)/v^{2-2/d} = c$. Here d is the degree of each graph in B, while $E(v)$ and v refers to the edges and vertices of the random graph. Then $\mathbf{P}_{v,E(v)}(A_k)$, $k = 0, 1, 2, \ldots$, is, asymptotically, a Poisson distribution*

$$\mathbf{P}_{v,E(v)}(A_k) \sim \lambda^k e^{-\lambda}/k!,$$

where we define $p = E(v)/\binom{v}{2}$ and

$$\lambda = \sum_{i=1}^{m} \binom{v}{v_i} B_i p^{E_i} (1-p)^{\binom{v_i}{2} - E_i}. \tag{1}$$

In particular,

$$\mathbf{P}_{v,E(v)}(A_0) \sim e^{-\lambda}. \tag{$*$}$$

λ is the asymptotic expected number of subgraphs isomorphic to some graph G_i in B. In the alternative model... of a random graph in which each edge is chosen independently according to a Bernoulli trial with probability p of success, λ is the precise expected number of subgraphs isomorphic to some graph G_i in B. As v increases, $\lambda \to \lambda^*$, where

$$\lambda^* = \sum_{i=1}^{m} B_i (2c)^{E_i}/v_i!, \tag{2}$$

and we have the precise statement

$$\lim_{v\to\infty} \mathbf{P}_{v,E(v)}(A_k) = (\lambda^*)^k e^{-\lambda^*}/k!. \text{''} \tag{$**$}$$

There are exactly 4 graphs with 4 labelled vertices that are isomorphic to K_{13}: any one of the 4 vertices may be chosen to be the central vertex of K_{13} and the remaining vertices are fixed. Therefore, with $m = i = 1$, $B_1 = 4$, $E_1 = 3$, $v_1 = 4$ and $k = 0$, $(**)$ and (2) of Theorem 2 imply (3) of the theorem to be proved, and $(*)$ and (1) of Theorem 2 imply (4) of the theorem to be proved. This completes the proof.

A graph G is defined to be an indifference graph [4] if and only if there exists a function from the set A of vertices into the real line such that for all x in y in A, $x\,I\,y$ if and only if $|f(x)-f(y)|\leqslant 1$. A graph G is defined to be a proper interval graph if and only if, for each x in A, there exists an interval $J(x)$ of the real line such that (1) holds and no one of these intervals is properly contained in another. The class of unit interval graphs is the same as the class of indifference graphs and the class of proper interval graphs [4, p. 144]. Thus our theorem also gives the asymptotic probabilities of these classes of graphs.

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