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Contractive inhomogeneous products of non-negative matrices

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1. *Introduction.* Hajnal (5) showed that under wide conditions a sequence of products $H(1,q) = M_q \ldots M_1$, $q = 1, 2, \ldots$, of square non-negative matrices M_q approaches a sequence of positive matrices of rank 1. We call a product $H(1,q)$ inhomogeneous if its factors $M_1, \ldots, M_q$ are not necessarily all equal to one another. When the matrices M_q are members of an 'ergodic set', and x and y are positive vectors, the projective distance $d(H(1,q)x, H(1,q)y)$ decays at least exponentially fast as q increases. An important condition on an ergodic set is that any product of g members from the set be a matrix in which all elements are (strictly) positive, where g is some fixed positive integer.

For several kinds of non-negative square matrices M_q which arise in biology and demography, $H(1,q)$ may fail to satisfy this requirement of positivity because every M_q is reducible. The purpose of this paper is to extend Hajnal's theory to describe some contractive sets of reducible non-negative matrices such that products $H(1,q)$ may not ever be positive. In some cases, products of these matrices are exponential contractions on the projective distance between positive initial vectors.

In this paper, we confine our attention to $n \times n$ non-negative non-zero matrices M which may be put in the normal or canonical form

$$M = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}, \tag{1}$$

possibly after applying the same permutation to rows and columns, where A is $n_1 \times n_1$, $1 \leqslant n_1 \leqslant n$, and irreducible. We define $n_1 = n$ to mean that the submatrices B, C and 0 are all null (i.e. contain no elements), so that $M = A$. C is $n_2 \times n_2$, where $n_1 + n_2 = n$, and may have zero, one, or several irreducible diagonal block submatrices. In general, B and C might be zero. The representation of a given M in the form (1) need not be unique. A square non-negative non-zero matrix which cannot be put in the form (1) is

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

although any square non-negative matrix with at least one positive element in each row can be put in the form (1) (Seneta (10), p. 13).

According to Loève ((9), pp. 367–369), Markov proved that if any non-negative matrix M is stochastic (all row sums equal to 1) and has a positive column (necessarily belonging to the submatrices A and B in (1)), then $M^q \to \mathbf{1}.\pi^T$ exponentially element-wise, where $\mathbf{1}$ is a column n-vector of 1's and π^T is a row n-vector with every element

π_i ~~positive~~ such that $\Sigma_i \pi_i = 1$. Bernstein's generalization of this result to inhomogeneous products of stochastic matrices each with a positive column (Seneta (10), p. 105) raises the suspicion that the positive column may be a key condition for an inhomogeneous product of reducible but not necessarily stochastic matrices to act contractively under the projective pseudo-metric on positive vectors. It turns out that a positive column is neither a necessary nor a sufficient condition for inhomogeneous products of reducible, non-stochastic matrices to act contractively.

Section 2 presents definitions and known results. Section 3 examines the contractive properties, individually and in inhomogeneous products, of non-negative matrices with at least one positive column. Section 4 presents the main results: an algebraic identity (9) for inhomogeneous products of reducible matrices and several sets of sufficient conditions for sets of reducible, non-stochastic, non-negative matrices to be contractive sets. Section 5 applies these results to matrices which arise in demography, genetics, and cell biology.

2. *Definitions and known results.* Where possible we follow the terminology of Hajnal (5). Matrices may be rectangular unless otherwise specified. A matrix A is subrectangular (Kaijser (6), p. 678) if $a_{i_1 j_1} a_{i_2 j_2} \neq 0$ implies that $a_{i_1 j_2} a_{i_2 j_1} \neq 0$. For any product AB of matrices A and B, it is tacitly assumed that A has as many columns as B has rows. A non-negative matrix (vector) is a matrix (vector) of real numbers all ≥ 0. A positive matrix (vector) is a matrix (vector) of real numbers all > 0. Vectors are columns unless transposed by ^{T}.

A row-allowable matrix is a non-negative matrix with at least one positive element in each row. A column-positive matrix is a non-negative matrix with at least one column which is a positive vector. A column-positive matrix is row-allowable.

A product of a finite number of row-allowable matrices is a row-allowable matrix. Moreover if A is row-allowable and of dimension $m \times n$ and x is a positive n-vector then Ax is a positive m-vector.

Hence if A is row-allowable and B is a column-positive matrix, then BA is column-positive and for every index i, $i = 1, \ldots, n$ such that the ith column $B^{(i)}$ of B is positive, the ith column $(AB)^{(i)}$ of AB is positive.

Given any two non-negative non-zero n-vectors $x = (x_i)$ and $y = (y_i)$ such that $x_i = 0$ if and only if $y_i = 0$, let $r_i = x_i/y_i$ when $y_i \neq 0$ and $r_i = 0$ otherwise. Define

$$d(x,y) = \ln[(\max_i r_i)/(\min_{i:r_i \neq 0} r_i)] = \max_{i,j:\, y_i x_j \neq 0} \ln(x_i y_j/(x_j y_i)).$$

This Hilbert projective distance d (defined on pairs of non-negative non-zero vectors with ~~positive elements~~ in corresponding positions) is a pseudo-metric, and $d(x,y) = 0$ if and only if $x = cy$ for some scalar $c > 0$. Bushell (3) gives elementary derivations of the principal properties of d, including the triangle inequality, in a general (Banach space) setting.

Let $\min^+(A)$ denote the smallest positive element of a matrix A and $\max(A)$ the largest positive element of $A = (a_{ij})$. Let $\|A\| = \max_i \Sigma_j |a_{ij}|$. Thus if $x > 0$ is a vector, $\|x\| = \max_i x_i$. Similarly let $\operatorname{mrs}(A) = \min_i \Sigma_j |a_{ij}|$; $\operatorname{mrs}(x) = \min_i x_i$ (mrs = minimal row sum). For a real scalar c, let $\operatorname{int}(c)$ denote the greatest integer less than or equal to c.

LEMMA 1 (essentially Hajnal (5)). *For any two matrices $A \geqslant 0$, $B \geqslant 0$ such that AB is defined,* (i) $\|AB\| \leqslant \|A\|\,\|B\|$ *and* (ii) $\text{mrs}(AB) \geqslant \text{mrs}(A)\,\text{mrs}(B)$. (iii) *If x and y are positive n-vectors and A is a row-allowable $m \times n$ matrix, $d(Ax, Ay) \leqslant d(x, y)$.*

Proof. (i) $\|\,.\,\|$ is a matrix norm. (ii) For every i,

$$\Sigma_k (AB)_{ik} = \Sigma_k \Sigma_j a_{ij} b_{jk} = \Sigma_j a_{ij} \Sigma_k b_{jk} \geqslant \text{mrs}(A)\,\text{mrs}(B).$$

So $\min_i \Sigma_k (AB)_{ik} \geqslant \text{mrs}(A)\,\text{mrs}(B)$. (iii) Apply (i) and (ii) first with $B = x$, then with $B = y$.

LEMMA 2 (*Bushell* (3)). *Let x and y be positive vectors. Let c_1 and c_2 be positive scalars. Then $d(x,y) = d(c_1 x, c_2 y)$.*

A square matrix $A \geqslant 0$ is defined as primitive if, for some positive integer k, $A^k > 0$. A square matrix $A \geqslant 0$ is defined as irreducible if, for every row i and column j, there exists a positive integer k which may depend on i and j such that $(A^k)_{ij} > 0$. A primitive matrix is irreducible.

LEMMA 3 (Brauer (2), p. 30). *If $A \geqslant 0$ of dimension $n \times n$ is irreducible and has at least one positive element on the main diagonal, then A^{2n-2} is positive.*

That the converse of Lemma 3 is false is demonstrated by a primitive 3×3 matrix A with zero main diagonal such that A^2 is positive:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Let ρM be the spectral radius (magnitude of the largest eigenvalue) of any square matrix M. Define 0 as the spectral radius of a null (0×0) matrix.

LEMMA 4 (Gantmacher (4), pp. 75, 69). *In the canonical form* (1) *of a non-negative square matrix M, $\rho M = \max(\rho A, \rho C)$.*

A $n \times n$ matrix M is semi-irreducible (Birkhoff and Varga (1), p. 359) when it can be put in the form, possibly after a permutation of rows and columns,

$$M = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix},$$

where A is an irreducible $n_1 \times n_1$ matrix and each row of B contains at least one non-zero element.

If a semi-irreducible matrix M is also non-negative and its A is primitive, then for q large enough, the first n_1 columns of M^q are positive. A column-positive matrix M is semi-irreducible if, in its normal form (1), $C = 0$.

Our main results rest on Hajnal's ((5), p. 527) concept of an ergodic set.

An ergodic set $G(g, R)$ with parameters g and R is a set of allowable $n \times n$ matrices A such that, for some integer g and a constant $R > 0$, any product of g matrices chosen from $G(g, R)$ is a positive matrix and for every A in $G(g, R)$

$$\min{}^{+}(A)/\max(A) > R. \tag{2}$$

Hajnal denotes R by ρ, which we have already reserved for the spectral radius function.

THEOREM A ((5), p. 527). *If $M_1, \ldots, M_q$ are any q matrices of order n belonging to some ergodic set $G(g, R)$, and $H(1,q) = M_q \ldots M_1$, then for any positive n-vectors x and y*

$$d(H(1,q)x, H(1,q)y) \leqslant d(x,y)[(1-\delta)/(1+\delta)]^{\mathrm{Int}(q/g)}, \quad (3)$$

where $\delta = R^g/n^{g-1}$.

(This expression for δ may not be the largest possible, but that given by Hajnal (p. 528), namely $\delta = R^g$, is incorrect. The derivation of R^g/n^{g-1} parallels exactly an argument used in proving Theorem 5 below.)

Extending a terminological suggestion of Hajnal (5), we propose two further definitions.

A contracting set S is a set of $n \times n$ matrices such that if x and y are any two positive n-vectors, then for any $\epsilon > 0$ there is an integer N (possibly depending on x and y) such that for all $q \geqslant N$, and for any sequence $M_1, \ldots, M_q, \ldots$ of matrices chosen from S, $d(H(1,q)x, H(1,q)y) < \epsilon$, where $H(1,q) = M_q \ldots M_1$.

A set S of $n \times n$ matrices is an exponentially contracting set if S is a contracting set such that for any positive n-vectors x and y, there exist positive constants $K < 1$ and D (with D possibly depending on x and y) such that for any partial products $H(1,q)$ of q arbitrary matrices from S, $d(H(1,q)x, H(1,q)y) \leqslant DK^q$.

Theorem A shows that an ergodic set is an exponentially contracting set.

3. *Column-positive matrices.* We now describe the effect on the projective distance between two positive vectors of a single column-positive matrix, of powers of a single column-positive matrix, and of various inhomogeneous products of column-positive matrices.

LEMMA 5. *Let x and y be positive vectors, $d(x,y) > 0$, A a column-positive matrix, $x^* = Ax$, $y^* = Ay$. Then $d(x^*, y^*) < d(x,y)$.*

Proof. Let $r_i^* = x_i^*/y_i^*$, $r^* = (r_i^*)$, $M = \max_i r_i$, $m = \min_i r_i$, $M^* = \max_i r_i^*$, $m^* = \min_i r_i^*$. As Hajnal (5) observed in proving the equivalent of Lemma 1, $m^* \geqslant m$ and $M^* \leqslant M$ and $r_i^* = \Sigma_j a_{ij} x_j / \Sigma_j a_{ij} y_j = \Sigma_j r_j a_{ij} y_j / \Sigma_j a_{ij} y_j$ so r_i^* is a weighted mean of the r_j with weights $p_{ij} = a_{ij} y_j / \Sigma_j a_{ij} y_j$. For every i, some but not all of the p_{ij} may be 0. With full generality, take column 1 of A to be positive so that $p_{i1} > 0$.

Let a be the index of a row such that $r_a^* = M^*$, b be the index of a row such that $r_b^* = m^*$.

$$M^* = r_a^* = p_{a1} r_1 + \ldots + p_{aj} r_j + \ldots,$$

$$m^* = r_b^* = p_{b1} r_1 + \ldots + p_{bj} r_j + \ldots,$$

where some or all of p_{aj}, p_{bj}, $j = 2, \ldots, n$ may be 0. If $M^* = M$ and $m^* = m$, then $a \neq b$. $r_i \leqslant M = M^*$ implies $r_1 = M$ and $r_i \geqslant m = m^*$ implies $r_1 = m$, a contradiction. So $m^* > m$ or $M^* < M$ or both. In any event, $M^* - m^* < M - m$, hence $M^*/m^* - 1 < M/m^* - m/m^* = (M/m - 1)(m/m^*) \leqslant M/m - 1$, so $M^*/m^* < M/m$ and $d(x^*, y^*) < d(x,y)$. Lemma 5 is proved.

THEOREM 1. *Let z be a real number satisfying $0 < z \leqslant 1$. Let $S(z)$ be the set of all column-positive matrices A such that* $\min^+(A)/\max(A) \geqslant z$. *Then for any positive vectors x, y such that $d(x,y) > 0$*

$$\sup_{A \in S(z)} \frac{d(Ax, Ay)}{d(x,y)} = t < 1.$$

(Since t depends on x and y as well as z, we shall write $t = t(x,y,z)$ when we need to be explicit.)

Proof. Let $S'(z)$ be the set of all column-positive matrices B such that $\max(B) \leqslant 1$ and $\min^+(B) \geqslant z$. Then $S'(z) \subset S(z)$, and for every A in $S(z)$, $A/\max(A)$ is in $S'(z)$. By Lemma 2,

$$t = \sup_{B \in S'(z)} \frac{d(Bx, By)}{d(x,y)}.$$

Now if each B is viewed as a point in Euclidean n^2-space, then $S'(z)$ is compact. Since $d(Bx, By)/d(x,y)$ is a continuous real function on $S'(z)$, it attains its supremum t at say B^* in $S(z)$. If it were true that $t = 1$, then $d(B^*x, B^*y) = d(x,y)$ would contradict Lemma 5. Theorem 1 is proved.

LEMMA 6. *Let z be a real number such that $0 < z \leqslant 1$. Let $X(z)$ be the set of all positive vectors x such that $\min^+(x)/\max(x) \geqslant z$. Then* $\operatorname{diam}(X(z)) = \sup_{x,y \in X(z)} d(x,y) = -2\ln z$.

Proof. Referring to the notation in the proof of Lemma 5, $M \leqslant z^{-1}$ and $m \geqslant z$ so $d(x,y) \leqslant -2\ln z$.

THEOREM 2. *Let z_1, z_3 be real numbers satisfying $0 < z_1, z_3 \leqslant 1$, and g a positive integer. Let $X(z_1)$ be the set of all positive vectors x such that* $\min^+(x)/\max(x) \geqslant z_1$. *Let $T(g, z_3)$ be any set of row-allowable matrices A such that* (i) $\min^+(A)/\max(A) \geqslant z_3$; *and* (ii) *a product of any g factors (including possible repetitions) from $T(g, z_3)$ is column-positive. Let $z_2 = (z_3/n)^g$ and $H(1,q) = A_q \ldots A_1$. Then uniformly for any x, y in $X(z_1)$ and any sequence $A_1, A_2, \ldots, A_q, \ldots$ of not necessarily distinct members of $T(g, z_3)$,*

$$d(H(1,q)\,x, H(1,q)\,y) \leqslant \ln K^*(q) \tag{4}$$

where $K^*(q) = K(\operatorname{int}(q/g))$, *and*

$$K(q+1) = [z_2 v(q) + (n-1)K(q)]/[z_2 v(q) + n - 1], \quad K(0) = z_1^{-2}, \tag{5}$$

$$v(q+1) = z_2 v(q)/[v(q) + n - 1], \quad v(0) = z_1. \tag{6}$$

Proof. $H(1,g)$ is in $S(z_2)$, which is the set of all column-positive matrices A such that $\min^+(A)/\max(A) \geqslant z_2$, though not every matrix B in $S(z_2)$ need be of the form $H(1,q)$. Let $x(0)$, $y(0)$ be in $X(z_1)$; $B(1), B(2), \ldots$, be a sequence of matrices in $S(z_2)$;

$$x(q) = B(q) \ldots B(1)\,x(0), \quad y(q) = B(q) \ldots B(1)\,y(0), \quad q = 1, 2, \ldots;$$

and $x_i(q)$ the ith element of $x(q)$. Let $r_i(q) = x_i(q)/y_i(q)$, $i = 1, \ldots, n$. Let $M(q) = \max_i r_i(q)$, $m(q) = \min_i r_i(q)$, $L(q) = M(q)/m(q)$. Then $d(x(q), y(q)) = \ln L(q)$. By Lemma 6, $L(0) \leqslant (w(0))^{-2} \leqslant z_1^{-2}$, where $w(q) = \min_{i,j} \min(x_i(q)/x_j(q),\ y_i(q)/y_j(q))$. We claim

$$L(q+1) \leqslant [z_2 w(q) + (n-1)L(q)]/[z_2 w(q) + n - 1], \tag{7}$$

and

$$w(q+1) \geqslant z_2 w(q)/[w(q) + n - 1] > 0. \tag{8}$$

Since we are interested in the effect of $B(q+1)$ on $x(q)$ and $y(q)$, abbreviate $B = B(q+1)$ and assume without loss of generality that column 1 of B is positive, i.e. $z_2 \leqslant b_{i1} \leqslant 1$, and $b_{ij} = 0$ or $z_2 \leqslant b_{ij} \leqslant 1$ for every element b_{ij} of B, $i = 1, \ldots, n; j = 2, \ldots, n$. $x(q)$ and $y(q)$ can be rescaled to be vectors whose least elements are not less than $w(q)$ and whose largest elements are 1.

If $L(q) = 1$, we are done. If $L(q) > 1$, there are 3 possibilities: (i) $r_1(q) = m(q)$; (ii) $r_1(q) = M(q)$; (iii) $m(q) < r_1(q) < M(q)$.

(i) Assuming $r_1(q) = m(q)$, let $M(q+1) = r_a(q+1)$ and $p_{ij} = b_{ij} y_j(q)/\Sigma_k b_{ik} y_k(q)$. Then $M(q+1) = \Sigma_j b_{aj} x_j(q)/\Sigma_k b_{ak} y_k(q) = \Sigma_j p_{aj} r_j(q) = p_{a1} m(q) + \Sigma_{j=2}^n p_{aj} r_j(q)$. To obtain an upper bound on $M(q+1)$, we minimize p_{a1} and maximize p_{aj} and $r_j(q)$, $j = 2, \ldots, n$. By choosing $(b_{a1}, \ldots, b_{an}) = (z_2, 1, 1, \ldots, 1)$, $y(q)^T = (w(q), 1, \ldots, 1)^T$, we have $z_2 w(q)/[z_2 w(q) + n - 1] \leqslant p_{ak} \leqslant [z_2 w(q) + n - 1]^{-1}$, $k = 1, \ldots, n$, though neither bound is sharp. If we further assume $r_j(q) = M(q), j = 2, \ldots, n$, then

$$M(q+1) \leqslant [z_2 w(q)\, m(q) + (n-1) M(q)]/[z_2 w(q) + n - 1].$$

Then $L(q+1) = M(q+1)/m(q+1) \leqslant M(q+1)/m(q)$ gives (7).

(ii) Assuming $r_1(q) = M(q)$, let $m(q+1) = r_b(q+1) = \Sigma_{j=1}^n b_{bj} x_j/\Sigma_k b_{bk} y_k$ so that $1/m(q+1) = \Sigma_{j=1}^n b_{bj} x_j (r_j(q))^{-1}/\Sigma_k b_{bk} x_k$. To obtain an upper bound on $1/m(q+1)$, we again minimize the weight of $(r_1(q))^{-1} = (M(q))^{-1}$ and maximize the remaining weights, assuming $(r_j(q))^{-1} = (m(q))^{-1}, j = 2, \ldots, n$, by choosing $(b_{b1}, \ldots, b_{bn}) = (z_2, 1, \ldots, 1)$ and $x(q)$ as $y(q)$ in case (i). Then $L(q+1) \leqslant M(q)/m(q+1) \leqslant M(q)\,[z_2 w(q)\,(M(q))^{-1} + (n-1)\,(m(q))^{-1}]/[z_2 w(q) + n - 1]$, again giving (7).

(iii) If $m(q) < r_1(q) < M(q)$, then, repeating previous arguments, $m(q+1) \geqslant [z_2 w(q) r_1(q) + (n-1) m(q)]/[z_2 w(q) + n - 1]$ while $M(q+1) \leqslant [z_2 w(q) r_1(q) + (n-1) M(q)]/[z_2 w(q) + n - 1]$. Hence $L(q+1) = M(q+1)/m(q+1) \leqslant [z_2 w(q) + (n-1) M(q)/r_1(q)]/[z_2 w(q) + (n-1) m(q)/r_1(q)] = f(r_1(q))$. By elementary calculus, $\partial f/\partial r_1(q) < 0$ for $r_1(q)$ in $[m(q), M(q)]$. Therefore setting $r_1(q) = m(q)$ only increases $f(r_1(q))$ and again gives (7).

The positive lower bound on the right in (8) is obtained by premultiplying the vector $y(q)^T = (w(q), 1, \ldots, 1)^T$ by the row $(z_2, 0, \ldots, 0)$ to obtain the numerator and by the row $(1, \ldots, 1)$ to obtain the denominator.

Then $L(q)$ is bounded above by the solution of the difference equation which results from replacing the inequality in (7) by equality. Since $L(q) > 1$, this solution in turn is bounded above by the solution obtained when $w(q)$ is taken to be as small as possible. A lower bound $v(q)$ on $w(q)$ is given by the solution of the difference equation which results from replacing the inequality in (8) by equality. This establishes the theorem for products of matrices from $T(g, z_3)$ taken in blocks of g at a time. Lemma 1 (iii) assures the result for all values of q.

The requirements in Theorem 2 of a bound (i) on the ratio of positive elements, analogous to condition (2), and the existence (ii) of a positive column do not guarantee that for a fixed square column-positive matrix A, $\{A\}$ is a contractive set. For example, if

$$A = \begin{pmatrix} 1 & 0{\cdot}25 & 0 \\ 0 & 0{\cdot}25 & 0 \\ 0 & 0{\cdot}5 & 1 \end{pmatrix},$$

$x^T = (0{\cdot}01,\ 0{\cdot}02,\ 0{\cdot}97)$ and $y^T = (0{\cdot}97,\ 0{\cdot}02,\ 0{\cdot}01)$, then $\lim_q A^q x = (5/300,\ 0,\ 295/300)^T = x^{*T}$ and $\lim_q A^q y = (293/300,\ 0,\ 7/300)^T = y^{*T}$. For any finite q, $d(A^q x, A^q y) \geqslant d(x^*, y^*) = \ln\,[(293\times 295)/(5\times 7)] = 7{\cdot}811799904 > 0$. In the notation of Theorem 2, we have $T(1,\ 0{\cdot}25) = \{A\}$, $n = 3$, $z_1 = 0{\cdot}01/0{\cdot}97$, and $z_2 = 0{\cdot}25/3$. Numerical iteration of (5) and (6) quickly (i.e. when $q = 6$) brings $\ln K(q)$ to the fixed point $\ln K(q) = 9{\cdot}148973963$. Numerical comparison shows that, as (4) asserts, $\ln K(q) \geqslant d(A^q x, A^q y)$ for every q, with equality only for $q = 0$.

LEMMA 7. *Let M be a column-positive matrix. Then in the canonical form* (1) *of M, A is also column-positive and primitive, B is column-positive, $\rho M > 0$, and $\rho A > 0$.*

Proof. Permutation of rows leaves a positive column positive. Every column-positive matrix (e.g. A) has a positive diagonal element and is primitive by Lemma 3. Finally, $\rho M \geqslant m_{ii}$ for all i.

It follows readily from Lemma 7 that if M is a column-positive matrix in normal form (1), then for all $k \geqslant 2n_1 - 2$, the first n_1 columns of M^k are positive.

THEOREM 3. *Let M be a column-positive square matrix in normal form* (1) *with $\rho A > \rho C$. (i) M has one positive eigenvector v corresponding to the positive eigenvalue $\rho A = \rho M$, and the modulus of every other eigenvalue of M is smaller than ρM. (ii) For any R, $0 < R < \infty$, and for any $\epsilon > 0$, there exists an integer q_0 such that for all $q \geqslant q_0$, and for any positive vectors x and y, if $d(x, v) \leqslant R$ and $d(y, v) \leqslant R$ then $d(M^q x, M^q y) < \epsilon$. (iii) $\{M\}$ is an exponentially contracting set.*

Proof. (i) Gantmacher ((4), p. 77, his theorem 6) proves the existence of v associated with ρM, since no square block submatrix on the diagonal of M is 'isolated' (in his language) other than A. Because A is primitive (by Lemma 3), no other eigenvalue of A or M has modulus equal to ρM by the Perron–Frobenius theorem. (ii) Let U_R be the set of positive vectors $\{w; \|w\| = 1,\ \epsilon/2 \leqslant d(w, v) \leqslant R\}$. U_R is compact. Let $t_M = \sup\{d(Mw, v)/d(w, v); w \in U_R\}$. Since U_R is compact, $t_M < 1$ by Lemma 5. So for all $w \in U_R$, $d(Mv, Mw) = d((\rho M)\, v, Mw) = d(v, Mw) \leqslant t_M d(v, w) \leqslant t_M R$. By iteration, $\sup\{d(v, M^q w); w \in U_R\} \leqslant t_M^q R < \epsilon/2$ for $q \geqslant q_0$. For such q, if $d(x, v) \leqslant R$ and $d(y, v) \leqslant R$, then $d(M^q x, v) < \epsilon/2$ and $d(M^q y, v) < \epsilon/2$ and, by the triangle inequality, $d(M^q x, M^q y) < \epsilon$. (iii) Let $R = \max(d(x, v), d(y, v))$ and apply part (ii).

COROLLARY 1 (Markov). *If M is a column-positive square stochastic matrix, then $\{M\}$ is an exponentially contracting set.*

Proof. In the normal form (1) of M, every row sum of A is 1 and every row sum of C is < 1. Hence $\rho A = 1$ and $\rho C < 1$. Theorem 3 applies.

COROLLARY 2. *For a fixed scalar z, $0 < z < 1$, and some positive vector v, let $S(z)$ be a set of column-positive matrices M such that (a) $\min^+(M)/\max(M) \geqslant z$; (b) in the normal form (1) of each M, $\rho A > \rho C$; (c) the positive eigenvector associated with ρM (by Theorem 3 (i)) is identical for all members of $S(z)$ and denoted by v. Let $\epsilon > 0$. (i) For any R, $0 < R < \infty$, there exists q_0 such that if $q \geqslant q_0$, $d(x, v) \leqslant R$, $d(y, v) \leqslant R$, then for all $M_1, \ldots, M_q \in S(z)$, $d(M_q \ldots M_1 x,\ M_q \ldots M_1 y) < \epsilon$. (ii) $S(z)$ is an exponentially contracting set.*

Proof. For any given positive vectors x, y, by Theorem 1 there exists $t(x,y,z) < 1$ such that if $M \in S(z)$, then $d(Mx, My)/d(x,y) \leqslant t(x,y,z)$. Define U_R as in the proof of Theorem 3(ii). Since $U_R \times U_R$ is compact, $t(z) = \sup\{t(x,y,z) : x, y \in U_R\} < 1$. Thus for all M in $S(z)$ and all w in U_R, $d(Mw, Mv) = d(Mw, v) \leqslant t(z)\, d(w,v)$ and the rest of the proof of Theorem 3(ii, iii) may be repeated to give Corollary 2.

4. *Sufficient conditions for reducible, non-stochastic matrices to be a contractive set.* Unfortunately, even if every matrix in a set of matrices satisfies the assumptions of Theorem 3, it is not true that inhomogeneous products of such matrices need satisfy either the assumptions or the conclusions of the theorem. For example, let

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0{\cdot}1 & 1 & 0 & 0 \\ 1 & 0{\cdot}5 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0{\cdot}1 & 1 \\ 1 & 0 & 0 & 0{\cdot}5 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0{\cdot}1 & 0{\cdot}5 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0{\cdot}1 & 0{\cdot}5 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix},$$

and $M = M_1 M_2$. Then M_1 and M_2 satisfy Theorem 3 separately, with $\rho M_1 = \rho M_2 = 1$, but M, with two eigenvalues equal to $\rho M > 1$, does not.

The possibility of generalizing Corollary 2 to inhomogeneous products of matrices significantly more general than those assumed in Corollary 2 is suggested by Theorem 4. The scalars a, b, c here correspond to measures of magnitude of the matrices A, B, C in the normal form (1).

THEOREM 4. *Let K_1 and K_2 be finite constants and let S be a set of 2×2 matrices M with finite elements of the form*

$$M = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$$

such that $a > 0$, $b > 0$, and (i) $0 \leqslant c/a \leqslant K_1 < 1$; (ii) $0 < K_2 \leqslant b/a$. Then S is an exponentially contracting set.

Proof. Let $H(1,q) = M_q \ldots M_1$. It is readily verified by induction that

$$H_{11}(1,q) = a_q \ldots a_1 > 0,$$
$$H_{12}(1,q) = 0,$$
$$H_{21}(1,q) = c_q \ldots c_2 b_1 + b_q a_{q-1} \ldots a_1 + \Sigma_{j=2}^{q-1} (c_q \ldots c_{j+1}) b_j (a_{j-1} \ldots a_1) > 0,$$
$$H_{22}(1,q) = c_q \ldots c_1 \geqslant 0.$$

Then $0 \leqslant H_{22}(1,q)/H_{21}(1,q) \leqslant c_q \ldots c_1/(b_q a_{q-1} \ldots a_1) \leqslant (c_q/b_q) K_1^{q-1}$ by (i). But (ii) implies $a_q/b_q \leqslant 1/K_2$. Multiplying by $c_q/a_q \leqslant K_1$ from (i) gives $c_q/b_q \leqslant K_1/K_2$ and hence $H_{22}(1,q)/H_{21}(1,q) \leqslant K_1^q/K_2$. Without loss of generality we may take $x^T = (1, x_2)$, $y^T = (1, y_2)$, $x_2 > y_2 > 0$. Then

$$\begin{aligned} d(H(1,q)\,x, H(1,q)\,y) &= \ln([H_{21}(1,q) + H_{22}(1,q)\,x_2]/[H_{21}(1,q) + H_{22}(1,q)\,y_2]) \\ &\leqslant \ln[1 + x_2 H_{22}(1,q)/H_{21}(1,q)] \leqslant (x_2/K_2) K_1^q. \end{aligned}$$

The assumptions of Theorem 4 describe an exponentially contracting set S that is not an ergodic set (since here $H_{12}(1,q) = 0$ for all q). Viewed as a set of points in Euclidean n^2-space, S need not even be closed or compact. E.g., if $S = \{M_q; q = 1, 2, \ldots\}$, where

$$M_q = \begin{pmatrix} 1 & 0 \\ 1 & (q+1)^{-1} \end{pmatrix},$$

the assumptions of Theorem 4 are met but S does not include $\lim_{q\to\infty} M_q$. Moreover, no matrix M in S is subrectangular as assumed in contraction theorems of Kaijser (6).

THEOREM 5. *Let $S = S(g, R, K_1, K_2, K_3)$ be a set of $n \times n$ non-negative matrices M, each of which has the same normal form (1), i.e. n_1 and n_2 are the same for all M in S. Let the collection of all the $n_1 \times n_1$ submatrices of members of S in the position corresponding to A be an ergodic set with parameters g and R. Let every submatrix corresponding to B be row-allowable. Suppose there exist constants K_1, K_2, and K_3 such that $0 \leqslant n_2 K_1 < 1$, $0 < K_2 \leqslant K_3 < \infty$, and for each M in S, (i) $0 \leqslant \max(C)/\min^+(A) \leqslant K_1$; (ii) $K_2 \leqslant \min^+(B)/\max(A)$; and (iii) $\max(B)/\min^+(A) \leqslant K_3$. Then for $q > g$, every product $H(1,q)$ of q arbitrary matrices from S has its first n_1 columns positive and S is an exponentially contracting set. The rate of contraction is a simple explicit function of the parameters of S.*

Proof. Let $M_1, \ldots, M_q, \ldots$ be an infinite sequence of matrices from S. For $p \leqslant q$, let $H(p,q) = M_q \ldots M_p$ be the 'backwards' product of the pth through the qth matrix from this sequence. Let $H_{11}(p,q)$ be the top left $n_1 \times n_1$ submatrix of $H(p,q)$, corresponding in position to the submatrix A of M. Similarly let $H_{12}(p,q)$ be the submatrix of $H(p,q)$ corresponding in position to the 0 submatrix of M, $H_{21}(p,q)$ be the submatrix of $H(p,q)$ corresponding to B in M, and $H_{22}(p,q)$ be the $n_2 \times n_2$ submatrix in the lower right corner of $H(p,q)$ corresponding to the submatrix C of M. It readily verified by induction that

$$H_{11}(1,q) = A_q \ldots A_1,$$

$$H_{12}(1,q) = 0,$$

$$H_{21}(1,q) = C_q \ldots C_2 B_1 + B_q A_{q-1} \ldots A_1 + \sum_{j=2}^{q-1} (C_q \ldots C_{j+1}) B_j (A_{j-1} \ldots A_1),$$

$$H_{22}(1,q) = C_q \ldots C_1.$$

Since the collection of all A submatrices is an ergodic set, $H_{11}(1,q) > 0$ for $q \geqslant g$. Since B_q is row-allowable, $B_q A_{q-1} \ldots A_1 > 0$ and $H_{21}(1,q) > 0$ for $q > g$. Thus the first n_1 columns of $H(1,q)$ are positive for $q > g$.

To show that S is an exponentially contracting set, choose positive n-vectors x and y. Let x^A be the n_1-vector containing the first n_1 elements of x and x^B be the n_2-vector containing the last n_2 elements of x; similarly let y be partitioned into y^A and y^B.

Now for any integer $Q \geqslant 1$ and any $q > Q$, $H_{11}(1,q) = A_q \ldots A_1 = A_q \cdots A_{Q+1} A_Q \cdots A_1$

$= H_{11}(Q+1,q)\,H_{11}(1,Q)$. Similarly $H_{22}(1,q) = H_{22}(Q+1,q)\,H_{22}(1,Q)$. With the convention that $C_q \dots C_j = I_{n_2\times n_2}$ if $j > q$ and $A_j \dots A_i = I_{n_1\times n_1}$ if $j < i$, we rewrite $H_{21}(1,q)$

$$= \sum_{j=1}^{Q} (C_q \dots C_{j+1}) B_j (A_{j-1} \dots A_1) + \sum_{j=Q+1}^{q} (C_q \dots C_{j+1}) B_j (A_{j-1} \dots A_1)$$

$$= C_q \dots C_{Q+1} \sum_{j=1}^{Q} (C_Q \dots C_{j+1}) B_j (A_{j-1} \dots A_1) + \left[\sum_{j=Q+1}^{q} (C_q \dots C_{j+1}) B_j (A_{j-1} \dots A_{Q+1})\right] \times A_Q \dots A_1$$

$$= H_{22}(Q+1,q)\,H_{21}(1,Q) + H_{21}(Q+1,q)\,H_{11}(1,Q).$$

Then

$$H(1,q)\,x = \begin{pmatrix} H_{11}(1,q) & 0 \\ H_{21}(1,q) & H_{22}(1,q) \end{pmatrix} \begin{pmatrix} x^A \\ x^B \end{pmatrix}$$

$$= \begin{pmatrix} H_{11}(Q+1,q) \\ H_{21}(Q+1,q) \end{pmatrix} H_{11}(1,Q)\,x^A + \begin{pmatrix} 0 \\ H_{22}(Q+1,q)\,[H_{21}(1,Q)\,x^A + H_{22}(1,Q)\,x^B] \end{pmatrix}. \tag{9}$$

Let $H_{+1}(Q+1,q)$ be the $n \times n_1$ submatrix containing the first n_1 columns of $H(Q+1,q)$, i.e. containing $H_{11}\ (Q+1,q)$ in the first n_1 rows, $H_{21}(Q+1,q)$ in the last n_2 rows. $H_{+1}(Q+1,q)$ is row-allowable. By the triangle inequality,

$$\begin{aligned} d(H(1,q)\,x, H(1,q)\,y) \leqslant\ & d(H(1,q)\,x, H_{+1}(Q+1,q)\,H_{11}(1,Q)\,x^A) \\ & + d(H_{+1}(Q+1,q)\,H_{11}(1,Q)\,x^A, H_{+1}(Q+1,q)\,H_{11}(1,Q)\,y^A) \\ & + d(H_{+1}(Q+1,q)\,H_{11}(1,Q)\,y^A, H(1,q)\,y). \end{aligned} \tag{10}$$

Since the set of all A submatrices is an ergodic set, by Theorem A, $d(H_{11}(1,Q)\,x^A$, $H_{11}(1,Q)\,y^A) \leqslant z^Q D_2'$, where $z = ([1 - R^g/n_1^{g-1}]/[1 + R^g/n_1^{g-1}])^{1/g} < 1$ and, for $z > 0$, $D_2' = d(x^A, y^A)/z$. If $z = 0$, the middle term on the right side of (10) vanishes and requires no further discussion. So we proceed assuming $z > 0$. By Lemma 1, $d(H_{+1}(Q+1,q)\ H_{11}(1,Q)\,x^A,\ H_{+1}(Q+1,q)\ H_{11}(1,Q)\,y^A) \leqslant z^Q D_2'$. Let F be the smallest integer which satisfies, for $z > 0$,

$$(n_1/R)\,(n_2 K_1)^F \leqslant z. \tag{11}$$

(Let $F = 0$ if $z = 0$.) For each q, choose $Q_1 = \operatorname{int}(q/(F+1))$. Then for $Q = Q_1$,

$$z^{Q_1} D_2' \leqslant [z^{1/(F+1)}]^q (D_2'/z) = [z^{1/(F+1)}]^q D_2, \text{ where } D_2 = d(x^A, y^A)/z^2.$$

For $i = 1, \dots, n$, let the scalar $r_i(x)$ be the quotient of the ith element of $H(1,q)\,x$ divided by the ith element of $H_{+1}(Q_1+1,\ q)\,H_{11}(1,Q_1)\,x^A$. From (9) it is evident that $r_i(x) = 1$ for $i = 1, \dots, n_1$ and $r_i(x) \geqslant 1$ for $i = n_1+1, \dots, n$. Thus $\min_i r_i = 1$. Letting j range from $n_1 + 1$ to n,

$$\begin{aligned} &(\max_i r_i(x))/(\min_i r_i(x)) = \max_j r_j(x) \\ &= 1 + \max_j (H_{22}(Q_1+1,q)\,[H_{21}(1,Q_1)\,x^A + H_{22}(1,Q_1)\,x^B])_j/(H_{21}(Q_1+1,q)\,H_{11}(1,Q_1)\,x^A)_j \\ &\leqslant 1 + \max_j \left[(C_q \dots C_{Q_1+1}) \left(\sum_{k=1}^{Q_1} C_{Q_1} \dots C_{k+1} B_k A_{k-1} \dots A_1\right) x^A\right]_j \\ &\qquad /[B_q A_{q-1} \dots A_{Q_1+1} A_{Q_1} \dots A_1 x^A]_j \\ &\qquad + \max_j [C_q \dots C_1 x^B]_j/[B_q A_{q-1} \dots A_1 x^A]_j. \end{aligned}$$

When $[B_q A_{q-1} \dots A_1 x^A]_j$ is written out as a multiple sum of products of scalars, at least one term must be positive since B_q is row-allowable, and this term must contain as factors one element of B_q, one element from each of $A_1, \dots, A_{q-1}$ and one element of x^A. As for the numerators, for each $k = 1, \dots, Q_1$, $[C_q \dots C_{k+1} B_k A_{k-1} \dots A_1 x^A]_j$ is the sum of at most $n_2^{q-k} n_1^k$ positive terms each containing $q+1$ scalar factors, of which $q-k$ are elements from C submatrices, 1 is an element from a B submatrix, $k-1$ are elements from A submatrices, and 1 is an element of x^A. Similarly, $[C_q \dots C_1 x^B]_j$ is the sum of at most n_2^q positive terms each containing $q+1$ scalar factors, of which q are elements from C submatrices and 1 is an element of x^B. Let $X = \max(x)/\min(x)$, $Y = \max(y)/\min(y)$. From (ii), $\max(A)/\min^+(B) \leqslant 1/K_2$; hence, multiplying by (i), $\max(C)/\min^+(B) \leqslant (K_1/K_2)(\min^+(A)/\max(A)) \leqslant K_1/K_2$. By (2), $1 \leqslant \max(A)/\min^+(A) < 1/R$. Then, using also (i) and (iii) and comparing terms factor by factor,

$$\left.\begin{aligned} \max_j r_j(x) &\leqslant 1 + \sum_{k=1}^{Q_1} n_2^{q-k} n_1^k (K_1/K_2) K_1^{q-1-k} K_3 (1/R)^{k-1} X + n_2^q (K_1/K_2) K_1^{q-1} X \\ &= 1 + (n_2 K_1)^q (X/K_2) \left[1 + K_3 R \sum_{k=1}^{Q_1} (n_1/(R n_2 K_1))^k\right] \\ &= 1 + (n_2 K_1)^q (X/K_2)[1 + K_3 R \beta(\beta^{Q_1} - 1)/(\beta - 1)], \\ &\leqslant 1 + (n_2 K_1)^q (X/K_2) + (X K_3 R\beta/(K_2(\beta-1)))(n_2 K_1)^q \beta^{Q_1}, \end{aligned}\right\} \tag{12}$$

where $\beta = n_1/(R n_2 K_1) > 1$. But

$$\begin{aligned} (n_2 K_1)^q \beta^{Q_1} &= (n_2 K_1)^q (n_1/(R n_2 K_1))^{\operatorname{int}(q/(F+1))} \leqslant (n_2 K_1)^q (n_1/(R n_2 K_1))^{q/(F+1)} \\ &= [(n_2 K_1)^{F+1} (n_1/(R n_2 K_1))]^{q/(F+1)} = [(n_1/R)(n_2 K_1)^F]^{q/(F+1)} \leqslant z^{q/(F+1)}; \end{aligned}$$

the last inequality follows from (11). Let $K_4 = \max(n_2 K_1, z^{1/(F+1)})$, and $D_1 = (X/K_2) \times (1 + K_3 R\beta/(\beta-1))$. Continuing (12), $\max_j r_j(x) \leqslant 1 + K_4^q D_1$ so

$$d(H(1,q)\,x, H_{+1}(Q_1+1,q)\,H_{11}(1,Q_1)\,x^A) = \ln(1 + \max_j r_j(x)) \leqslant K_4^q D_1.$$

By an exactly parallel calculation with x replaced by y and X by Y,

$$d(H(1,q)\,y, H_{+1}(Q_1+1,q)\,H_{11}(1,Q_1)\,y^A) \leqslant K_4^q D_3.$$

Returning to (10), we have

$$d(H(1,q)\,x, H(1,q)\,y) \leqslant K_4^q (D_1 + D_2 + D_3).$$

This proves the theorem.

Without loss of generality, one may suppose that in each matrix M in the set S considered in Theorem 5, every element is divided by $\min^+(A)$, so that $\min^+(A)$ becomes equal to 1. Then, since A is row-allowable by definition of an ergodic set, $\rho A \geqslant 1$. But $\rho C \leqslant n_2 \max(C) \leqslant n_2 K_1 < 1$ from (i), so $\rho C < \rho A$. Since B is row-allowable, some power of M has the first n_1 columns positive. Hence the conclusions of Theorem 3 apply to each separate matrix of S. But it is *not* assumed in Theorem 5 that any M in S is column-positive.

THEOREM 6. *Let S' be a set of $n \times n$ row-allowable matrices M' of the form*

$$M' = \begin{pmatrix} A' & 0 \\ B' & C' \end{pmatrix}$$

where A' is $n_1 \times n_1$, $n_1 \geqslant 1$, C' is $n_2 \times n_2$, $n_2 \geqslant 0$, $n_1 + n_2 = n$. Let $\{A'; M' \in S'\}$ be an ergodic set. Let $(B')_{1,n_1} = (M')_{n_1+1,n_1} > 0$; i.e., we do not require that B' be row-allowable, only that its upper right corner be positive. Let C' be strictly lower triangular with a positive subdiagonal; i.e. $(C')_{ij} = 0$ for $i \leqslant j$; $(C')_{i+1,i} > 0$ for $i = 1, \ldots, n_2 - 1$; $(C')_{i,j} \geqslant 0$ for $i > j+1$. Let $H(1,q) = M'_q \ldots M'_1$, where M'_j are arbitrary matrices from S' and $H_{11}(1,q)$, $H_{21}(1,q)$ and $H_{22}(1,q)$ be the submatrices of $H(1,q)$ corresponding in position respectively to A', B' and C' in M'. Then for $q \geqslant n_2$, $H_{22}(1,q) = 0$ (so that $H(1,q)$ is semi-irreducible), $H_{21}(1,q)$ is column-positive, and S' is an exponentially contracting set. The rate of contraction is not slower than the rate of the ergodic set of A' submatrices.

Proof. Matrix multiplication shows that the number of positive elements in the n_1th column of $H_{21}(1,q)$ increases to n_2, and the number of positive elements in every column of $H_{22}(1,q)$ decreases to 0, as q increases from 1 to n_2. Consequently for $q \geqslant 1$, $H_{22}(q+1, q+n_2) = 0$. Therefore, using the purely algebraic identity (9), for any positive vectors x and y, $H(1,q+n_2)x = H_{+1}(q+1,\ q+n_2)H_{11}(1,q)x^{A'}$ and $d(H(1,q+n_2)x,$ $H(1,q+n_2)y) \leqslant d(H_{11}(1,q)x^{A'}, H_{11}(1,q)y^{A'})$. This upper bound decreases exponentially as q increases.

The so-called Leslie or 'projection' matrices used in the demography of age-structured populations are a special case of the matrices considered in Theorem 6, which may therefore be viewed as a generalization of weak ergodic theorems in demography (e.g. Lange (8), his Lemma 7). No quantitative restrictions other than finiteness are imposed on the elements of B' and C'.

The duals of all results in sections 3 and 4 are obtained by exchanging row for column, left for right and forward products for backward products. Let a square matrix $A \geqslant 0$ have ith row and jth column both positive. If $i = j$ then A^2 is positive. If $i \neq j$, no power of A need be positive in general, as shown by M in Theorem 4.

5. *Applications.* Matrices like M_1 to M_5 below motivated the theory just developed. These generalize to $n \times n$ matrices. Subject to the quantitative restrictions on the elements of the matrices assumed in section 4, sets of matrices with each of these patterns of positive elements can be exponentially contracting sets.

$$M_1 = \begin{pmatrix} + & 0 & 0 \\ + & + & 0 \\ 0 & + & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} + & 0 & 0 \\ + & + & 0 \\ 0 & + & + \end{pmatrix}, \quad M_3 = \begin{pmatrix} + & + & 0 \\ 0 & + & 0 \\ 0 & + & + \end{pmatrix},$$

$$M_4 = \begin{pmatrix} + & 0 & 0 \\ + & 0 & 0 \\ 0 & + & 0 \end{pmatrix}, \quad M_5 = \begin{pmatrix} + & + & 0 \\ + & 0 & 0 \\ 0 & + & 0 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 0 & + & + \\ + & 0 & 0 \\ 0 & + & 0 \end{pmatrix}.$$

The + signs indicate positive numbers. In applications, each positive element m_{ij} models a flow from a category labelled j to a category labelled i. Products of a sufficiently large number of matrices of the form M_i are column-positive but not positive, for any single $i = 1, 2, \ldots, 5$. Products of a sufficiently large number of matrices like M_6 are positive. Each M_i is in normal form (1) except for M_3; for the other five, the irreducible submatrices A_i are of order 1, 1, 1, 2 and 3, respectively.

M_1 is a linear formalization of a compartmental model for the ontogeny of blood-forming cells (Till, McCulloch and Siminovitch (12)). Compartment 1 represents stem cells, compartment 2 early differentiated cells, and compartment 3 specialized descendants. Cells in the first 2 compartments, but not the third, can proliferate.

M_2 models a 3-stage educational system in which individuals either remain in their present state or are promoted at the end of each time period (Stone (11)).

Both M_1 and M_2 model the progress of patients in a 3-state disease. In model M_1, death or recovery from state 3 are inevitable; in M_2, they are not.

M_3 models the course from one (discrete) generation to the next of the proportions of individuals of genotypes AA, Aa, and aa, allowing for differential fertility, viability, and gametic selection (Karlin (7), where the transpose of M_3, more customary in Markov chain theory, is used). Column 2 gives the distribution of AA, Aa, and aa offspring of parents of Aa genotype. The numerical example A given after the proof of Theorem 2 fails to meet the assumption of Theorem 3, which in this example reduces to $m_{22} > \max(m_{11}, m_{33})$ and $m_{12} m_{32} > 0$.

M_4, M_5 and M_6 are all special cases of the demographic projection matrix for a closed, single-sex population with 3 age groups (young, middle, old). The elements of the first row describe age-specific fertility, those of the subdiagonal age-specific survival. M_6 falls within the scope of Hajnal's (5) theory.

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