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Chebyshev and Grüss inequalities for real rectangular matrices ☆

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ABSTRACT

Quadratic ordering of rectangular real matrices implies Chebyshev's inequality: If $A_1, \ldots, A_r$ are $m \times n$ real matrices and $B_1, \ldots, B_r$ are $n \times q$ real matrices such that, for all i, j with $1 \leqslant i, j \leqslant r$, elementwise $(A_i - A_j)(B_i - B_j) \geqslant 0^{m\times q}$, then for any real $p_j \geqslant 0$, $j = 1, \ldots, r$, $\sum_j p_j = 1$, elementwise $(\sum_j p_j A_j)(\sum_j p_j B_j) \leqslant \sum_j p_j A_j B_j$. Further, linear ordering of rectangular real matrices implies Grüss's inequality: If, elementwise, $A_j \leqslant A_{j+1}$, $j = 1, \ldots, r-1$ and elementwise $B_j \leqslant B_{j+1}$, $j = 1, \ldots, r-1$ then elementwise $\sum_j p_j A_j B_j - (\sum_j p_j A_j)(\sum_j p_j B_j) \leqslant \frac{1}{4}(A_r - A_1)(B_r - B_1)$. The bounds are sharp. These inequalities lead to inequalities for the spectral radius of nonnegative matrices. Linear ordering and quadratic ordering are equivalent for real scalars but not for real matrices.

1. Introduction and definitions

The inequalities of Chebyshev [6, p. 43, Theorem 43], [11, p. 76, Eq. (5.8)] and Grüss [5,11, p. 119] have been generalized from sequences of real numbers to square, complex, Hermitian positive definite or semidefinite matrices and orthonormal families of vectors in real or complex inner product spaces [1–3,7–10]. Here we generalize these inequalities to rectangular real matrices and derive inequalities involving the spectral radius of nonnegative matrices.

Let $\mathbb{N} = \{1, 2, \ldots\}$ be the natural numbers, $m, n, q, r, s \in \mathbb{N}$. Let $\mathbb{R} = (-\infty, +\infty)$ be the real line and $\mathbb{R}^{m\times n}$ the set of real matrices of size $m \times n$, i.e., with m rows and n columns. Let

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$\mathbb{P}^r = \{p = (p_1, \ldots, p_r) \mid 0 \leqslant p_j \leqslant 1,\ j = 1, \ldots, r,\ \sum_{j=1}^r p_j = 1\}$ be the set of r-dimensional probability vectors. The $m \times n$ matrix with all elements equal to $c \in \mathbb{R}$ is written $c^{m\times n}$. For any two matrices $A_1,\ A_2 \in \mathbb{R}^{m\times n}$, $A_1 \leqslant A_2$ means that elementwise $0^{m\times n} \leqslant A_2 - A_1$. Whereas $\leqslant$ is a total ordering of the set of real numbers, it is a partial ordering of the set of real matrices with 2 or more elements. For $A_1,\ A_2 \in \mathbb{R}^{m\times n}$, $A_1 < A_2$ means that $A_1 \leqslant A_2$ and $A_1 \neq A_2$. Matrix $A \in \mathbb{R}^{m\times n}$ is positive if elementwise $0^{m\times n}$ is strictly less than A and is nonnegative if $0^{m\times n} \leqslant A$.

Let $\mathcal{V} = \{(A_j, B_j) \mid A_j \in \mathbb{R}^{m\times n},\ B_j \in \mathbb{R}^{n\times q},\ j = 1, \ldots, r\}$ be a set of r pairs of real matrices with each pair labeled by j. Assume henceforth that $r > 1$. Let $\mathcal{A} = \{A_j \in \mathbb{R}^{m\times n}\}$ be the set of first components of the pairs in $\mathcal{V}$, and let $\mathcal{B} = \{B_j \in \mathbb{R}^{n\times q}\}$ be the set of second components of the pairs in $\mathcal{V}$, each labeled as in $\mathcal{V}$. We say $\mathcal{A}$, $\mathcal{B}$, $\mathcal{V}$ are positive when every matrix in them is positive and similarly for nonnegative. For any two scalars $a, b \in \mathbb{R}$, define $\mathcal{V}(a, b) = \{(A_j + a^{m\times n}, B_j + b^{n\times q}) \mid (A_j, B_j) \in \mathcal{V}, j = 1, \ldots, r\}$. Clearly, for every $\mathcal{V}$, there exist $a, b \in \mathbb{R}$ such that every element of every matrix in $\mathcal{V}(a, b)$ is positive.

Definition 1. The set $\mathcal{V}$ is *linearly ordered* when there exists a permutation σ of $\{1, \ldots, r\}$ such that $A_{\sigma(j)} \leqslant A_{\sigma(j+1)}$ and $B_{\sigma(j)} \leqslant B_{\sigma(j+1)}$ for $j = 1, \ldots, r - 1$.

When $\mathcal{V}$ is linearly ordered, it entails no loss of generality to permute the subscripts of both components by the same permutation so that $A_1 \leqslant \cdots \leqslant A_r$ and $B_1 \leqslant \cdots \leqslant B_r$. Such a relabeling will be assumed.

Definition 2. The set $\mathcal{V}$ is *quadratically ordered* when, for all i, j such that $1 \leqslant i, j \leqslant r$, $(A_i - A_j)(B_i - B_j) \geqslant 0^{m\times q}$.

When $\mathcal{V}$ is quadratically ordered, it remains quadratically ordered if any permutation is applied equally to the labels of the matrices in both $\mathcal{A}$ and $\mathcal{B}$. In the scalar case, $m = n = q = 1$, $\mathcal{V}$ is linearly ordered if and only if $\mathcal{V}$ is quadratically ordered [6, p 43]. This equivalence fails in general.

Example 1. Let $m = 1,\ n = 2,\ q = 1,\ r = 2$ with $A_1 = (+1, -1),\ A_2 = (0, 0),\ B_j = A_j^T,\ j = 1, 2$. (Superscript ^{T} denotes the matrix transpose.) Then $(A_1 - A_2)(B_1 - B_2) = A_1B_1 = 2 > 0$ so $\mathcal{V}$ is quadratically ordered but not linearly ordered.

Definition 3. Let $\max(\mathcal{A})$ be the $m \times n$ matrix such that the element in row g and column h of $\max(\mathcal{A})$ is the maximum of the elements in row g and column h of the matrices $A_i \in \mathcal{A},\ i = 1, \ldots, r$. Define $\min(\mathcal{A}),\ \max(\mathcal{B}),\ \min(\mathcal{B})$ analogously.

When $\mathcal{V}$ is linearly ordered, then $\max(\mathcal{A}) = A_r \in \mathcal{A},\ \min(\mathcal{A}) = A_1 \in \mathcal{A}$, $\max(\mathcal{B}) = B_r \in \mathcal{B},\ \min(\mathcal{B}) = B_1 \in \mathcal{B}$. When $\mathcal{V}$ is quadratically ordered, these inclusions need not hold. In Example 1, $\max(\mathcal{A}) = (1, 0),\ \min(\mathcal{A}) = (0, -1)$ and neither $\max(\mathcal{A})$ nor $\min(\mathcal{A})$ is an element of $\mathcal{A}$.

2. Main results and proofs

Theorem 1. *Let $\mathcal{V} = \{(A_j, B_j) \mid A_j \in \mathbb{R}^{m\times n},\ B_j \in \mathbb{R}^{n\times q},\ j = 1, \ldots, r\}$ be a set of r pairs of real matrices, $r > 1$, and $p \in \mathbb{P}^r$. Let*

$$D(\mathcal{V}, p) = \sum_{j=1}^{r} p_j A_j B_j - \left(\sum_{j=1}^{r} p_j A_j\right)\left(\sum_{j=1}^{r} p_j B_j\right) \tag{1}$$

and

$$U(\mathcal{V}) = \frac{1}{4}\left(\max(\mathcal{A}) - \min(\mathcal{A})\right)\left(\max(\mathcal{B}) - \min(\mathcal{B})\right). \tag{2}$$

If $\mathcal{V}$ is quadratically ordered, then (Chebyshev's inequality)

$$\forall p \in \mathbb{P}^r, \quad 0^{m\times q} \leqslant D(\mathcal{V}, p), \tag{3}$$

with equality if there exists $1 \leqslant j \leqslant r$ such that $p_j = 1$, and (Grüss's inequality)

$$\forall p \in \mathbb{P}^r, \quad D(\mathcal{V}, p) \leqslant U(\mathcal{V}), \tag{4}$$

with equality in (3) and (4) if all A_j, $j = 1, \ldots, r$ are equal or all B_j, $j = 1, \ldots, r$ are equal.

Proof. If $\mathcal{V}$ is quadratically ordered, then for any $i \neq j$, $(A_j - A_i)(B_j - B_i) \geqslant 0^{m\times q}$ and $p_i p_j (A_j - A_i)(B_j - B_i) \geqslant 0^{m\times q}$. Summing over all pairs i, j, we have, for any $p \in \mathbb{P}^r$,

$$0^{m\times q} \leqslant \frac{1}{2} \sum_{1\leqslant i,j\leqslant r} p_i p_j (A_j - A_i)(B_j - B_i) \tag{5}$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{1\leqslant i,j\leqslant r} p_i p_j (A_i B_i + A_j B_j - A_i B_j - A_j B_i) \\
&= \frac{1}{2} \sum_{1\leqslant i,j\leqslant r} p_i p_j (A_i B_i + A_j B_j) - \frac{1}{2} \sum_{1\leqslant i,j\leqslant r} p_i p_j (A_i B_j + A_j B_i) \\
&= \frac{1}{2} \left(\sum_{1\leqslant i\leqslant r} p_i A_i B_i + \sum_{1\leqslant j\leqslant r} p_j A_j B_j \right) \\
&\quad - \frac{1}{2} \left[\left(\sum_{1\leqslant i\leqslant r} p_i A_i \right) \left(\sum_{1\leqslant j\leqslant r} p_j B_j \right) + \left(\sum_{1\leqslant i\leqslant r} p_i B_i \right) \left(\sum_{1\leqslant j\leqslant r} p_j A_j \right) \right] \\
&= \sum_{j=1}^{r} p_j A_j B_j - \left(\sum_{j=1}^{r} p_j A_j \right) \left(\sum_{j=1}^{r} p_j B_j \right) = D(\mathcal{V}, p).
\end{aligned} \tag{6}$$

This proof of (3) follows the proofs in [6, p. 43] and [11, pp. 77-78] for scalars.

The summation in (5) includes r terms with $i = j$. Such terms must be $0^{m\times q}$, so when any $p_j = 1$, then $p_j^2 = 1$ and $D(\mathcal{V}, p) = 0^{m\times q}$.

We may also write, using (5) and the last line of (6)

$$D(\mathcal{V}, p) = \sum_{1\leqslant i<j\leqslant r} p_i p_j (A_j - A_i)(B_j - B_i). \tag{7}$$

For all $i < j$, $0^{m\times q} \leqslant p_i p_j (A_j - A_i)(B_j - B_i) \leqslant p_i p_j (\max(\mathcal{A}) - \min(\mathcal{A}))(\max(\mathcal{B}) - \min(\mathcal{B}))$. Let D' be the sum obtained from (7) by replacing the summand $p_1 p_r (A_r - A_1)(B_r - B_1)$ with $p_1 p_r (\max(\mathcal{A}) - \min(\mathcal{A}))(\max(\mathcal{B}) - \min(\mathcal{B}))$. Then $D(\mathcal{V}, p) \leqslant D'$. We maximize D' over all $p \in \mathbb{P}^r$ by assigning all probability mass to $i = 1$, $j = r$, and putting 0 probability mass on any other couple $i < j$, so that $D' = p_1 p_r (\max(\mathcal{A}) - \min(\mathcal{A}))(\max(\mathcal{B}) - \min(\mathcal{B}))$, where $p_r = 1 - p_1$. Then the maximum of $p_1 p_r$ is attained when $p_1 = p_r = 1/2$, so that $p_1 p_r = 1/4$. This proves (4). $\square$

Lemma 1. *If $\mathcal{V}$ is linearly ordered, then $\mathcal{V}$ is quadratically ordered. If $\mathcal{V}$ is quadratically ordered and $n = 1$ and every two elements of $\mathcal{A}$ are distinct and every two elements of $\mathcal{B}$ are distinct, then $\mathcal{V}$ is linearly ordered.*

Proof. Let $\mathcal{V}$ be linearly ordered. By transitivity, for all i, j, if $i \leqslant j$ then $A_i \leqslant A_j$ and $B_i \leqslant B_j$ and therefore $A_j - A_i \geqslant 0^{m\times n}$ and $B_j - B_i \geqslant 0^{n\times q}$. Because the product of two nonnegative matrices is nonnegative, $i \leqslant j$ implies $(A_j - A_i)(B_j - B_i) \geqslant 0^{m\times q}$. Because $(A_i - A_j)(B_i - B_j) = (A_j - A_i)(B_j - B_i)$ we have $(A_i - A_j)(B_i - B_j) \geqslant 0$ whether $i \leqslant j$ or $j \leqslant i$. Hence $\mathcal{V}$ is quadratically ordered.

Now assume $\mathcal{V}$ is quadratically ordered and $n = 1$ and every two elements of $\mathcal{A}$ are distinct and every two elements of $\mathcal{B}$ are distinct. Each A_i is a column m-vector, $m \geqslant 1$, with elements a_{ig}, $g = 1, \dots, m$. Each B_i is a row q-vector, $q \geqslant 1$, with elements b_{ih}, $h = 1, \dots, q$. The product $(A_i - A_j)(B_i - B_j) \geqslant 0$ is a rank-one matrix with element $(a_{ig} - a_{jg})(b_{ih} - b_{jh}) \geqslant 0$, $g = 1, \dots, m$, $h = 1, \dots, q$ in row g and column h. We shall prove that these conditions imply that $A_i \leqslant A_j$, $B_i \leqslant B_j$ or $A_j \leqslant A_i$, $B_j \leqslant B_i$, i.e., that (possibly after a simultaneous permutation of the labels of $\mathcal{A}$ and $\mathcal{B}$) $\mathcal{V}$ is linearly ordered. It suffices to consider $i = 1$, $j = 2$.

We are given that $(a_{2g} - a_{1g})(b_{2h} - b_{1h}) \geqslant 0$, $g = 1, \dots, m$, $h = 1, \dots, q$ and that $A_1 \neq A_2$, $B_1 \neq B_2$. Then for at least one value of g, say g', $a_{2g'} - a_{1g'} > 0$ or $a_{2g'} - a_{1g'} < 0$ and for at least one value of h, say h', $b_{2h'} - b_{1h'} > 0$ or $b_{2h'} - b_{1h'} < 0$. If $a_{2g'} - a_{1g'} > 0$, then necessarily $b_{2h'} - b_{1h'} > 0$ because $(a_{2g'} - a_{1g'})(b_{2h'} - b_{1h'}) \geqslant 0$. In this case, for no $g \neq g'$ could we have $a_{2g'} - a_{1g'} < 0$ because then, for such a g, $(a_{2g} - a_{1g})(b_{2h'} - b_{1h'}) < 0$. Hence $a_{2g} - a_{1g} \geqslant 0$ for all $g = 1, \dots, m$ and by a similar argument $b_{2h} - b_{1h} \geqslant 0$ for all $h = 1, \dots, q$. Consequently, $A_2 \geqslant A_1$, $B_2 \geqslant B_1$. In the second case, if $a_{2g'} - a_{1g'} < 0$, then by the same argument $A_2 - A_1 \leqslant 0$ and $B_2 - B_1 \leqslant 0$. This proves that the relation $\leqslant$ orders every two elements of $\mathcal{V}$. Hence, possibly after a permutation of labels, $\mathcal{V}$ is linearly ordered. □

Example 2. In Example 1, where $n > 1$, $\mathcal{V}$ is quadratically ordered but not linearly ordered even though every two elements of $\mathcal{A}$ are distinct and every two elements of $\mathcal{B}$ are distinct.

This example also shows that equality is not guaranteed to hold in (4) above nor in (9) below if there exists $1 \leqslant j \leqslant r$ such that $p_j = 1$. By (3), $D(\mathcal{V}, p) = 0$ whenever some $p_j = 1$ but here $\frac{1}{4}(\max(\mathcal{A}) - \min(\mathcal{A}))(\max(\mathcal{B}) - \min(\mathcal{B})) = \frac{1}{4}\left((1,1)(1,1)^T\right) = \frac{1}{2}$.

Example 3. To see that $\mathcal{V}$ being quadratically ordered need not imply that $\mathcal{V}$ must be linearly ordered if $n = 1$ while two elements of $\mathcal{A}$ or two elements of $\mathcal{B}$ are identical, let $A_1 = A_2 = 0^{2\times 1}$, $B_1 = (+1, -1)$, $B_2 = (0, 0)$. Then $(A_1 - A_2)(B_1 - B_2) = 0^{2\times 2}$ so $\mathcal{V}$ is quadratically ordered but not linearly ordered.

Example 4. The assumption that every two elements of $\mathcal{A}$ are distinct and every two elements of $\mathcal{B}$ are distinct is stronger than the assumption that every two elements of $\mathcal{V}$ are distinct. Example 3 shows that $\mathcal{V}$ may be quadratically ordered but not linearly ordered under the weaker assumption that for every $i \neq j$, the pairs (A_i, B_i) and (A_j, B_j) are distinct, even when $n = 1$. In this example, $(A_1, B_1) \neq (A_2, B_2)$ because $B_1 \neq B_2$.

Corollary 1. *If $\mathcal{V}$ is linearly ordered, then Chebyshev's inequality* (3) *holds and Grüss's inequality* (4) *takes the form*

$$\forall p \in \mathbb{P}^r, \quad D(\mathcal{V}, p) \leqslant \frac{1}{4}(A_r - A_1)(B_r - B_1). \tag{8}$$

Moreover, $\inf\{D(\mathcal{V}, p) \mid p \in \mathbb{P}^r\} = 0^{m\times q}$ *and* $\sup\{D(\mathcal{V}, p) \mid p \in \mathbb{P}^r\} = \frac{1}{4}(A_r - A_1)(B_r - B_1)$*, i.e., the lower and upper bounds cannot be improved.*

Proof. If $\mathcal{V}$ is linearly ordered, then it is quadratically ordered so (3) holds. In (7), for every $i < j$, $0^{m\times q} \leqslant (A_j - A_i)(B_j - B_i) \leqslant (A_r - A_1)(B_r - B_1)$, so $\sup\{D(\mathcal{V}, p) \mid p \in \mathbb{P}^r\}$ is attained by maximizing $p_1 p_r$ and putting 0 probability mass on any other summand. The maximum of $p_1 p_r$, 1/4, is attained when $p_1 = p_r = 1/2$. □

Corollary 2. *If $\mathcal{V}$ is quadratically ordered and $n = 1$ and every two elements of $\mathcal{A}$ are distinct and every two elements of $\mathcal{B}$ are distinct, then* (3) *and* (8) *hold.*

Proof. By Lemma 1, $\mathcal{V}$ is linearly ordered. Apply Corollary 1. □

Proposition 1. *For any $a, b \in \mathbb{R}$, $\mathcal{V}$ is linearly (alternatively, quadratically) ordered if and only if $\mathcal{V}(a, b)$ is linearly (alternatively, quadratically) ordered, and $D(\mathcal{V}, p) = D(\mathcal{V}(a, b), p)$.*

Proof. For $A_i, A_j \in \mathcal{A}$, we have $A_i \leqslant A_j$ if and only if, for every $a \in \mathbb{R}$, $A_i + a^{m\times n} \leqslant A_j + a^{m\times n}$. Hence $\mathcal{V}$ is linearly ordered if and only if every $\mathcal{V}(a, b)$ is. Also, if $a, b \in \mathbb{R}, A_i, A_j \in \mathcal{A}$, $B_i, B_j \in \mathcal{B}$, then $(A_i - A_j)(B_i - B_j) = ((A_i + a^{m\times n}) - (A_j + a^{m\times n}))((B_i + b^{n\times q}) - (B_j + b^{n\times q}))$. Hence $\mathcal{V}$ is quadratically ordered if and only if every $\mathcal{V}(a, b)$ is and (7) implies $D(\mathcal{V}, p) = D(\mathcal{V}(a, b), p)$. $\square$

Corollary 3. *Let $\rho(\cdot)$ be the spectral radius (maximum magnitude of the eigenvalues) of a square matrix argument. If $m = q$ and $\mathcal{V}$ is quadratically ordered, then*

$$\forall p \in \mathbb{P}^r, \quad \rho(D(\mathcal{V}, p)) \leqslant \rho(U(\mathcal{V})). \tag{9}$$

Equality holds in (9), and both sides equal 0, if all A_j, $j = 1, \ldots, r$ are equal or all B_j, $j = 1, \ldots, r$ are equal. If $\mathcal{V}$ is quadratically ordered and also nonnegative, then

$$\forall p \in \mathbb{P}^r, \quad \rho\left(\left(\sum_{j=1}^{r} p_j A_j\right)\left(\sum_{j=1}^{r} p_j B_j\right)\right) \leqslant \rho\left(\sum_{j=1}^{r} p_j A_j B_j\right). \tag{10}$$

Equality holds in (10) if there exists $1 \leqslant j \leqslant r$ such that $p_j = 1$ or all A_j, $j = 1, \ldots, r$ are equal or all B_j, $j = 1, \ldots, r$ are equal.

Proof. By (3) and (4), $D(\mathcal{V}, p) \geqslant 0^{m\times m}$ and $U(\mathcal{V}) \geqslant 0^{m\times m}$. The spectral radius, here the Perron-Frobenius root, is nondecreasing when any element of its nonnegative square matrix argument increases [4, vol. 2, p. 57]. Apply this fact to (4) to get (9) and to (3) to get (10). $\square$

3. Generalizations to products of more than two matrices

Hardy *et al.* [6, p. 44] offered as "an immediate deduction" from Chebyshev's inequality the generalization that $(\sum_j p_j a_j^t)^{1/t}(\sum_j p_j b_j^t)^{1/t} \cdots (\sum_j p_j l_j^t)^{1/t} < (\sum_j p_j a_j^t b_j^t \cdots l_j^t)^{1/t}$ if $t > 0$ and the vectors $a = (a_j), b = (b_j), \ldots, l = (l_j)$ "are all similarly ordered". But their definition [6, p. 43] of what it means for a pair of vectors a, b to be "similarly ordered," namely, $(a_i - a_j)(b_i - b_j) \geqslant 0$ for all i, j, would not generalize usefully to, for example, $(a_i - a_j)(b_i - b_j)(c_i - c_j) \geqslant 0$ for all i, j, because $(a_i - a_j)(b_i - b_j)(c_i - c_j) = -(a_j - a_i)(b_j - b_i)(c_j - c_i)$. Both $(a_i - a_j)(b_i - b_j)(c_i - c_j) \geqslant 0$ and $(a_j - a_i)(b_j - b_i)(c_j - c_i) \geqslant 0$ would imply both are 0.

When $t = 1$ (which makes Chebyshev's inequality valid for real scalars, not only nonnegative scalars [6, p. 43 footnote]), it is not sufficient for the proof of the generalization that every pair of vectors be linearly ordered in the sense of Definition 1. For example, if $a = (1, 2)$, $b = (-2, +1)$, $c = (-2, +1)$, then a, b, c are all in nondecreasing order but $b_1 c_1 = 4 > b_2 c_2 = 1$ so the elementwise product vector $(b_j c_j)_{j=1}^2$ is ordered oppositely from the vector a. Assuming nonnegativity avoids this problem.

Definition 4. Let $\mathcal{V} = \{(A_j, B_j, C_j) \mid A_j \in \mathbb{R}^{m\times n},\ B_j \in \mathbb{R}^{n\times q},\ C_j \in \mathbb{R}^{q\times s},\ j = 1, \ldots, r\}$ be a set of r triples of matrices with each triple labeled by j. $\mathcal{V}$ is *similarly ordered* if some simultaneous permutation of the labels produces $A_1 \leqslant \cdots \leqslant A_r$ and $B_1 \leqslant \cdots \leqslant B_r$ and $C_1 \leqslant \cdots \leqslant C_r$. Define $\mathcal{C} = \{C_j,\ j = 1, \ldots, r \mid (A_j, B_j, C_j) \in \mathcal{V}\}$.

Theorem 2. *Let $\mathcal{V}$ be a similarly ordered set of r triples of nonnegative matrices, $r > 1$, and $p \in \mathbb{P}^r$. Let*

$$D(\mathcal{V}, p) = \sum_{j=1}^{r} p_j A_j B_j C_j - \left(\sum_{j=1}^{r} p_j A_j\right)\left(\sum_{j=1}^{r} p_j B_j\right)\left(\sum_{j=1}^{r} p_j C_j\right) \tag{11}$$

and

$$U(\mathcal{V}) = \frac{1}{4}\left(A_r - A_1\right)\left(B_r - B_1\right)\left(C_r - C_1\right). \tag{12}$$

Then (Chebyshev's inequality)

$$\forall p \in \mathbb{P}^r, \quad 0^{m\times q} \leqslant D(\mathcal{V}, p), \tag{13}$$

with equality if there exists $1 \leqslant j \leqslant r$ *such that* $p_j = 1$*, and (Grüss's inequality)*

$$\forall p \in \mathbb{P}^r, \quad D(\mathcal{V}, p) \leqslant U(\mathcal{V}), \tag{14}$$

with equality in (13) and (14) if, for at least two of the three sets $\mathcal{A}, \mathcal{B}, \mathcal{C}$*, all the matrices in each set equal the other matrices in that set.*

Proof. Because (by assumption) $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are similarly ordered and the matrices in each set are nonnegative, we have $B_1C_1 \leqslant \cdots \leqslant B_rC_r$. By Theorem 1,

$$\sum_{j=1}^{r} p_jA_j(B_jC_j) \geqslant \left(\sum_{j=1}^{r} p_jA_j\right)\left(\sum_{j=1}^{r} p_jB_jC_j\right) \geqslant \left(\sum_{j=1}^{r} p_jA_j\right)\left(\sum_{j=1}^{r} p_jB_j\right)\left(\sum_{j=1}^{r} p_jC_j\right).$$

This proves (13). The proof of (14) is the same as the proof of (4). The conditions for equality are straightforward, as before. □

Corollary 4. *If* $\mathcal{V}$ *is a similarly ordered set of r triples of nonnegative matrices,* $r > 1,\ m = s$*, and* $p \in \mathbb{P}^r$*, then*

$$\forall p \in \mathbb{P}^r, \quad \rho(D(\mathcal{V}, p)) \leqslant \rho(U(\mathcal{V})) \tag{15}$$

and

$$\forall p \in \mathbb{P}^r, \quad \rho\left(\left(\sum_{j=1}^{r} p_jA_j\right)\left(\sum_{j=1}^{r} p_jB_j\right)\left(\sum_{j=1}^{r} p_jC_j\right)\right) \leqslant \rho\left(\sum_{j=1}^{r} p_jA_jB_jC_j\right). \tag{16}$$

Equality holds in (15) and (16) if there exists $1 \leqslant j \leqslant r$ *such that* $p_j = 1$ *or if, for at least two of the three sets* $\mathcal{A}, \mathcal{B}, \mathcal{C}$*, all the matrices in each set equal the other matrices in that set.*

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