Pythagoras in a Box

Joel E. Cohen

A diagram of the Pythagorean theorem displays a right triangle with squares constructed on each leg (of lengths a and b) and the hypotenuse (of length c). (See figure 1.) Arthur Carter, a versatile artist, businessman, and journalist, wrapped a bronze relief sculpture of the diagram of the Pythagorean theorem in a box: He surrounded it by the smallest rectangle that bounded all points in the diagram and that had at least one of its sides coincident with at least one of the sides of the squares (figure 2). In Carter's wrapping, two adjacent sides of the bounding rectangle coincided with one side of each of the squares on the legs, while the other two sides of the rectangle touched corners of the square on the hypotenuse.

A different way to wrap the Pythagorean diagram in a box (figure 3) is to make one side of the bounding rectangle coincide with one side of the square on the hypotenuse, while each other side of the rectangle touches one corner of one of the squares on the legs. I propose to call this way of wrapping the diagram of the Pythagorean theorem the *Retrac wrapping*, by contrast with the Carter wrapping.

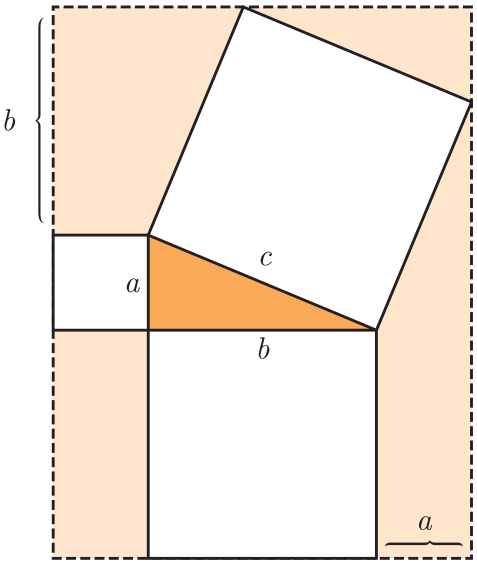

Figure 1: Diagram of the Pythagorean theorem, wrapped in a box by the Carter wrapping.

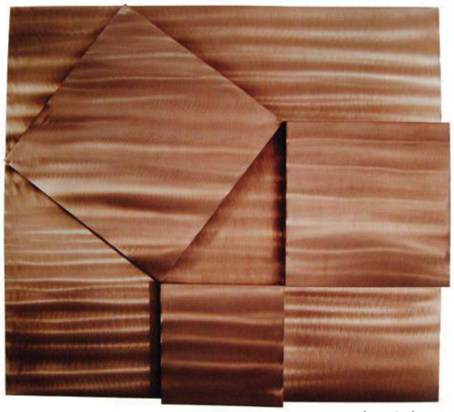

Figure 2: Photograph of Arthur Carter's *Mathematika II*, 2008, bronze, 30 × 33 in.

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Suppose the cost of a wrapping is proportional to the area of the bounding rectangle. For a given right triangle, which wrapping is less expensive? Although the question is entirely elementary, the answer is not obvious from inspection of the diagrams.

Let the two legs of the right triangle have positive lengths a and b and the hypotenuse have length

$$c = \sqrt{a^2 + b^2}.$$

Without loss of generality, we assume $a \le b$.

(Digression: The problem could be reduced to a one-parameter problem because the comparison of wrappings is independent of the scale of the right triangle, so we could require $c = 1$. We would then have a right triangle with legs of length a and $\sqrt{1-a^2}$. The assumption $a \le b$ would then be equivalent to $a \le 1/\sqrt{2}$. A one-parameter approach would gain in economy and lose in transparency and intuitive accessibility. We continue to call the sides a, b, and c.)

In the Carter wrapping (figure 1), the width of the rectangle, reading along the bottom side from left to right, is

$$W_C = a + b + a = 2a + b.$$

The height of the rectangle, reading along the left side from bottom to top, is

$$H_C = b + a + b = a + 2b.$$

Hence, the area of the Carter wrapping is

$$A_C = (2a+b)(a+2b) = 2(a^2+b^2) + 5ab.$$

This expression has a simple interpretation. The first term is the summed area of all three squares, and the second term is 10 times the area of the right triangle. It is easy to draw and count all 10 of these right triangles.

The calculations for the Retrac wrapping use extensively the elementary fact that if a triangle similar

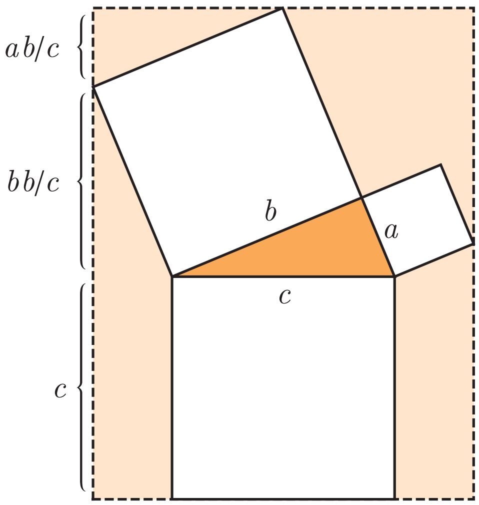

Figure 3: Diagram of the Pythagorean theorem, wrapped in a box by the Retrac wrapping.

to the given right triangle has hypotenuse x, then its shorter leg has length ax/c and its longer leg has length bx/c. In the Retrac wrapping, the width of the rectangle, reading along the bottom side from left to right, is

$$W_R = \frac{ab}{c} + c + \frac{ab}{c} = c + \frac{2ab}{c}.$$

The height of the rectangle, reading along the left side from bottom to top, is

$$H_R = c + \frac{b^2}{c} + \frac{ab}{c}.$$

(Interesting exercise: Calculate the width using the top side of the rectangle and the height using the right side. Verify that the answers are the same as those using the bottom and left sides, respectively.) Hence, the area of the Retrac wrapping is

$$A_R = (c + 2ab/c)(c + b^2/c + ab/c).$$

The difference between the Carter area and the Retrac area is

$$\begin{aligned}\Delta A &= A_C - A_R \\ &= \frac{2ab(c^2 - b(a+b))}{c^2} - b^2 + c^2 \\ &= \frac{2a^3(a+b)}{a^2+b^2} - a^2.\end{aligned}$$

Since $b \geq a$, we will write $b = a + \varepsilon$, with $\varepsilon \geq 0$. Then

$$\Delta A = \frac{a^2(2a^2 - \varepsilon^2)}{2a^2 + 2a\varepsilon + \varepsilon^2}.$$

The denominator is positive. The numerator will be positive, and hence ΔA will be positive, if and only if

$$\varepsilon < a\sqrt{2},$$

that is, if and only if $b < a(1+\sqrt{2})$. For example, if $b = a$, then $\Delta A = a^2$, while if $b = 3a$, then $\Delta A = -a^2/5$, and obviously if

$$b = a(1+\sqrt{2}),$$

then $\Delta A = 0$. Who would have guessed?

In Carter's sculpture, by my measurement of the photograph,

$$b/a \approx 1.318 < 1 + \sqrt{2} \approx 2.414.$$

Carter could have used less bronze to wrap his diagram of the Pythagorean theorem if he had used the Retrac wrapping. But perhaps economizing on bronze was not his purpose. Anyway, if he had used the more economical wrapping, it would not have been called the Retrac wrapping.

Both the Carter and the Retrac wrappings have greater area (and cost in bronze) than the convex hull of the diagram, equivalent to shrink-wrapping around the vertices of the three squares. What is the area of the convex hull? ■

I thank Florence Davis for sending me *Arthur Carter: Orthogonals*, the U.S. National Science Foundation for grant EF-1038337, Priscilla K. Rogerson for assistance, and the family of William T. Golden for hospitality during this work.

Reference

Charles A. Riley and Alfred Friedland, *Arthur Carter: Orthogonals* (New York: 80 Washington Square East Galleries 2011).

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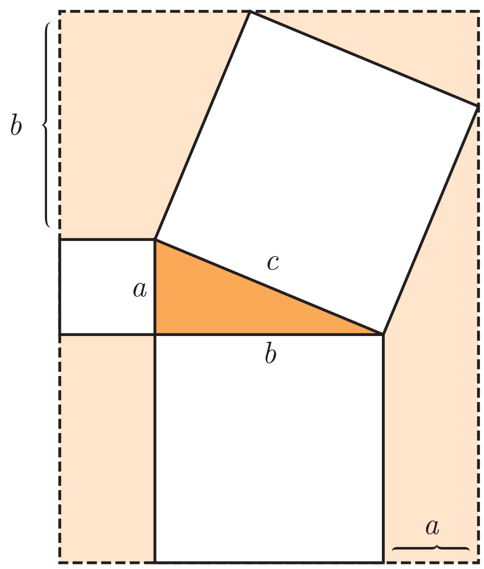
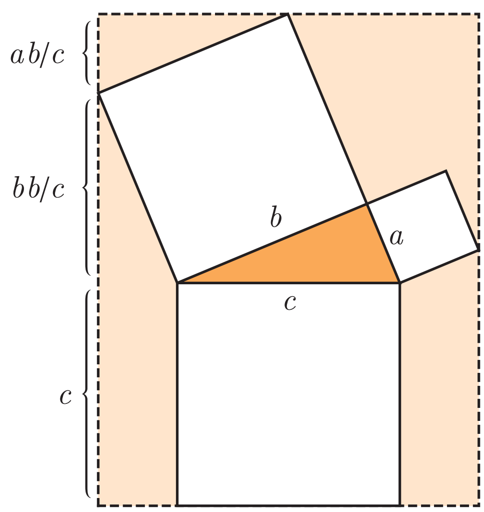
April's Zip-Line problem was connected to the article "Pythagoras in a Box" by Joel E. Cohen. The article compares two methods of inscribing a Pythagorean theorem diagram with rectangles, the Carter wrapping and the Retrac wrapping, as depicted below. Sometimes the area of the Carter wrapping is larger than the area of the Retrac wrapping, and sometimes it is smaller, depending on the dimensions of the right triangle.



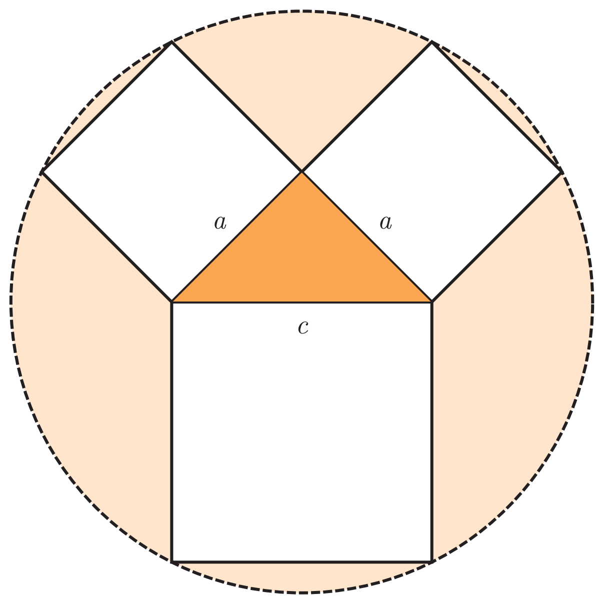
Carter wrapping



Retrac wrapping

Problem 276, That's a Wrap!, asked if it is possible for a *circle* containing a Pythagorean theorem diagram to have smaller area than either the Carter wrapping or the Retrac wrapping.

A correct solution to this problem was received from Michael Woltermann, who used an isosceles right triangle to show that, yes, it is possible for the circle to have smaller area than either the Carter or the Retrac wrapping.



Here, $c = \sqrt{2}a$, and the center of the circle is at the midpoint of the hypotenuse of the triangle. This means that the radius of the circle is $(\sqrt{10}/2)a$, so the area is $(5\pi/2)a^2$, which is less than $8a^2$. You can check that the Retrac wrapping has area $8a^2$ and the Carter wrapping has area $9a^2$.