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Competitive ternary interactions and relative entropy of solutions

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Abstract. For conservative linear systems (finite-state Markov processes in discrete or continuous time), the relative entropy of two distinct trajectories is a monotonically decreasing function of time. These results naturally raise the question whether distinct trajectories of nonlinear conservative systems also display monotonically decreasing relative entropy. For binary interacting Lotka–Volterra systems with anti-symmetry, the relative entropy oscillates under the motion. The main new result of this paper is that, for ternary interacting Lotka–Volterra systems with anti-symmetry, the relative entropy of two distinct trajectories is a monotonically decreasing function of time near equilibrium. Far from equilibrium, distinct trajectories of ternary Lotka–Volterra systems with anti-symmetry need not have monotonically decreasing relative entropy.

1. Introduction

Classical Lotka–Volterra equations (Lotka 1925, Volterra 1931) model pairwise interactions of individuals and, by extension, pairwise interactions of species. For example, let $p_i(t)$ be the fraction of individuals who belong to species i at time t. Kimura (1958) and Mather (1969) studied the model

$$\frac{\mathrm{d}}{\mathrm{d}t}p_i(t) = p_i(t)\sum_{j=1}^{m} a_{ij}p_j(t) \qquad \text{for} \quad t \geqslant t_0 \quad \text{where} \quad a_{ij}+a_{ji}=0$$

$$p_i(t_0) > 0 \qquad \sum_{i=1}^{m} p_i(t_0) = 1 \qquad \text{for} \quad i, j = 1, 2, \ldots, m\,.$$

These are quadratic differential equations because products of $p_i(t)$ appear on the right. Quadratic differential equations have been analysed using non-associative algebras by Markus (1960), McKean (1966), and Kaplan and Yorke (1979). By an analogy with the kinetic theory of gases, Itoh (1971, 1973, 1975, 1979, 1981) derived these Lotka–Volterra equations from a model of random collisions of particles of different species and used non-associative algebra to analyse the equations. Under the assumption of anti-symmetry $a_{ij} = -a_{ji}$, a well known important characteristic of these equations is that there exists a potential function that is conserved under the motion of the system (Kerner 1957, 1959, Goel *et al* 1971). It follows that if the initial condition of the system differs from equilibrium, then the system never approaches equilibrium.

Binary interactions may not be sufficient to model all situations of biological interest. At high population densities, three, four or more individuals may interact as for the Boltzmann equation for higher densities (Cohen 1973, Sengers 1973). As Mather (1969) stated, a plant may feel the effects of competition from a number of other individuals growing at various distances from it and interacting with one another in their effects on it. Models with ternary interactions have been investigated at least since Hutchinson (1947); Goel *et al* (1971, pp 266–9) review many other generalizations. Itoh (1975, 1981) analysed a differential equations model with ternary interactions using non-associative algebra. He proved that a certain Lyapunov function (given explicitly in theorem 1 below) increases until the system attains equilibrium. For the corresponding model with only binary interactions, the same Lyapunov function is invariant with respect to time. Thus the term that represents ternary interactions makes a qualitative difference to the model's behaviour and justifies, from the mathematical point of view, the study of models with ternary and higher-order interactions. A simulation study of competing species in which individuals are located on a regular lattice (Tainaka 1988, Tainaka and Itoh 1991) shows a stability that could be explained by the mathematical results on ternary and higher-order interactions.

From the empirical point of view, if an increase in the density of many interacting species were observed to lead to an increase in the stability of the size of the interacting populations, the difference between models with binary interactions and the models with ternary interactions might provide one explanation. Of course, one would have to investigate and exclude alternative explanations, such as a possible loss of the exact anti-symmetry condition as a result of increased population density.

For conservative linear systems (finite-state Markov processes in discrete or continuous time), it has been known for a long time (Moran 1961, Morimoto 1963, Csiszár 1963) that the relative entropy of two distinct trajectories is a monotonically decreasing function of time. Cohen *et al* (1993a, b) give the following improvement. Let p and r be two m-element probability vectors with positive elements. The relative entropy $H(p,r)$ of p and r is defined by $H(p,r) = \sum_i p_i \log(p_i/r_i)$. If A is an $n \times m$ matrix with elements $a_{ij} \geqslant 0$ such that $\sum_i a_{ij} = 1$, $j = 1, \ldots, m$, then $H(Ap, Ar) \leqslant \bar{\alpha}(A) H(p,r)$, where $\bar{\alpha}(A) = (\frac{1}{2}) \max_{j,k} \sum_{i=1}^{n} |a_{ij} - a_{ik}| \leqslant 1$. An analogous result for Markov processes in continuous time bounds $\mathrm{d}\log H(p(t), r(t))/\mathrm{d}t$ below zero.

These results naturally raise the question whether distinct trajectories of nonlinear conservative systems also display monotonically decreasing relative entropy. For *binary* Lotka–Volterra systems with anti-symmetry, the answer is no, because the relative entropy oscillates under the motion (Kerner 1957, 1959, Goel *et al* 1971). The main new result of this paper (theorem 2) is that, for *ternary* Lotka–Volterra systems with anti-symmetry, the answer is yes near equilibrium. Far from equilibrium, distinct trajectories of ternary Lotka–Volterra systems with anti-symmetry need not have monotonically decreasing relative entropy.

2. Random collision model for competitive interaction and non-associative algebra A^m for a Lotka–Volterra equation

We consider the following random collision model.

(i) There are m species labelled $1, 2, \ldots, m$ whose numbers of particles are, at time t, $n_1(t), n_2(t), \ldots, n_m(t)$, respectively, with $n_1(t) + n_2(t) + \cdots + n_m(t) = n$, where n is constant.

(ii) Each particle collides with another particle on average $\mathrm{d}t$ times per time length $\mathrm{d}t$.

(iii) Each particle is in a chaotic bath of particles. Each colliding pair is equally likely to be chosen.

(iv) For $i, j = 1, 2, \ldots, m$, by a collision, a particle of species i and a particle of species j become two particles of species i with probability $\frac{1}{2} + a_{ij}$, and two particles of species j with probability $\frac{1}{2} - a_{ij}$, where $a_{ij} = -a_{ji}$ and $-\frac{1}{2} \leqslant a_{ij} \leqslant \frac{1}{2}$.

When n is sufficiently large, we can derive equations in the following way.

Each of $((n_j(t)/n)\mathrm{d}t)n_i(t)$ particles of species i collides with a particle of species j and remains in species i with probability $\frac{1}{2}+a_{ij}$. Each of $((n_i(t)/n)\mathrm{d}t)n_j(t)$ particles of species j collides with a particle of species i and changes to species i with probability $\frac{1}{2}+a_{ij}$. So we have

$$n_i(t+\mathrm{d}t) = n_i(t)(1-\mathrm{d}t) + \frac{n_i(t)\,\mathrm{d}t}{n}\left\{\sum_{j=1}^{m}\left(\tfrac{1}{2}+a_{ij}\right)n_j\right\} + \left\{\sum_{j=1}^{m}\left(\tfrac{1}{2}+a_{ij}\right)\frac{n_j(t)\,\mathrm{d}t}{n}\right\}n_i(t) \tag{1}$$

$$\mathrm{d}n_i(t) = n_i(t+\mathrm{d}t) - n_i(t).$$

Put $n_i(t)/n = p_i(t)$, then we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}p_i(t) &= p_i(t)\left(\sum_{j=1}^{m}\left(\tfrac{1}{2}+a_{ij}\right)p_j(t)\right) + \left(\sum_{j=1}^{m}\left(\tfrac{1}{2}+a_{ij}\right)p_j(t)\right)p_i(t) - p_i(t)\\ &= 2p_i(t)\left(\sum_{j=1}^{m}a_{ij}p_j(t)\right) \qquad \text{for} \quad i = 1, 2, \ldots, m.\end{aligned} \tag{2}$$

We define the following non-associative algebra A^m to extend our discussion to ternary and higher-order interactions.

Definition. The non-associative algebra A^m is defined as follows:

(I) $A^m = \left\{\sum_{i=1}^{m} x_i E_i \mid x_i \in R, i = 1, 2, \ldots, m\right\}$

is an m-dimensional linear space over a field R (which here is always the real numbers) which is generated by linearly independent elements $E_i, i = 1, 2, \ldots, m$.

(II) The products of the basis elements are defined as

$$E_i \circ E_j = \left(\tfrac{1}{2}+a_{ij}\right)E_i + \left(\tfrac{1}{2}+a_{ji}\right)E_j$$

where

$$a_{ij} = -a_{ji} \qquad \text{and} \qquad -\tfrac{1}{2} \leqslant a_{ij} \leqslant \tfrac{1}{2}.$$

(III) The product $x \circ y$ of two elements

$$x = \sum_{i=1}^{m} x_i E_i,\ y = \sum_{j=1}^{m} y_j E_j \in A^m$$

is defined as

$$\sum_{i=1}^{m} x_i E_i \circ \sum_{j=1}^{m} y_j E_j = \sum_{i,j=1}^{m} x_i y_j (E_i \circ E_j).$$

A^m has the following properties.

Property 1. We see from the above definition that

$$E_i \circ E_j = E_j \circ E_i \qquad E_i \circ E_i = E_i .$$

Thus the algebra is commutative.

Hereafter we write the ith component of $x \in A^m$ as x_i.

Property 2. For $x, y \in A^m$, we have

$$\sum_{i=1}^{m}(x \circ y)_i = \sum_{i,j=1}^{m} x_i y_j = \left(\sum_{i=1}^{m} x_i\right)\left(\sum_{j=1}^{m} y_j\right).$$

Using the non-associative algebra A^m, equation (2) is expressed by

$$\frac{\mathrm{d}}{\mathrm{d}t} p(t) = p(t) \circ p(t) - p(t) . \tag{3}$$

The system with ternary interactions is represented by

$$\frac{\mathrm{d}}{\mathrm{d}t} p(t) = k_1(p(t) \circ p(t) - p(t)) + k_2((p(t) \circ p(t)) \circ p(t) - p(t)) \qquad \text{for} \quad p(t) \in A^m . \tag{4}$$

Using property 2, it is obvious that $\sum_i((p(t) \circ p(t)) \circ p(t))_i = \sum_i(p(t) \circ p(t))_i = \sum_i(p(t))_i = 1$, hence the binary system and the ternary system are conservative, i.e. $\sum_i \mathrm{d}p_i(t)/\mathrm{d}t = 0$. At first glance, definition (3) of the system with ternary interactions appears to have a strange asymmetry in it. Why does the right-hand side not contain an additional term $k_3(p(t) \circ (p(t) \circ p(t)) - p(t))$? The answer is that, because the algebra A^m is commutative, the term that is apparently missing would be exactly equivalent to the existing term with leading coefficient k_2.

We assume

$$(p_1(t_0), p_2(t_0), \ldots, p_m(t_0)) \in B^m = \left\{ p \,\middle|\, \sum_{i=1}^{m} p_i = 1, \, p_i > 0 \qquad \text{for} \quad i = 1, 2, \ldots, m \right\}.$$

Theorem 1. (Itoh 1981). Let there exist a unique $q \in B^m$ which satisfies $q \circ q - q = 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{m} q_i \log p_i(t) = 2k_2 \sum_{i=1}^{m} q_i \left(\sum_{j=1}^{m} a_{ij} p_j(t) \right)^2 \geqslant 0 \tag{5}$$

if $(p_1(t_0), p_2(t_0), \ldots, p_m(t_0)) \in B^m$.

Remark. It is easy to prove that if there exists a unique $q \in B^m$ such that $q \circ q - q = 0$, then $m \neq 2$. Also, if $m = 3$, then A must have the form

$$A = \begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix}$$

where a, b, and c are all positive or all negative. In this case, $q_1 = c/S$, $q_2 = b/S$, $q_3 = a/S$, where $S = a + b + c$.

3. Relative entropy near the equilibrium

Theorem 2. For q as defined in theorem 1, let $p(t) \in B$ and $r(t) \in B$, $t \geqslant t_0$, be two distinct solutions of the ternary system (4), $p(t_0) \neq r(t_0)$. Let $p = q + \delta, r = q + \varepsilon$. If $\max_i |\delta_i/q_i|$ and $\max_i |\varepsilon_i/q_i|$ are sufficiently small, then

$$\frac{\mathrm{d}H(p(t), r(t))}{\mathrm{d}t} \approx -2k_2 \sum_i q_i \left(\sum_j a_{ij}(p_j - r_j) \right)^2 < 0 . \tag{6}$$

Proof. We have

$$\begin{aligned} H(p,r) &= \sum_i (q_i + \delta_i) \log \frac{p_i}{r_i} \\ &= \sum_i (q_i \log p_i + \delta_i \log p_i - q_i \log r_i - \delta_i \log r_i) \end{aligned} \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} H(p,r) = \frac{\mathrm{d}}{\mathrm{d}t} \sum_i (q_i \log p_i - q_i \log r_i) + \frac{\mathrm{d}}{\mathrm{d}t} \sum_i (\delta_i \log p_i - \delta_i \log r_i) . \tag{8}$$

Put

$$\alpha_i = \frac{(p \circ p - p)_i}{p_i} = \sum_j a_{ij} p_j \qquad \gamma_i = \frac{(r \circ r - r)_i}{r_i} = \sum_j a_{ij} r_j .$$

We have from (5)

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_i (q_i \log p_i - q_i \log r_i) \equiv k_2 I_1 = 2k_2 \sum_i (q_i \alpha_i^2 - q_i \gamma_i^2) . \tag{9}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t} \delta = k_1(p \circ p - p) + k_2((p \circ p) \circ p - p) \tag{10}$$

we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \sum_i (\delta_i \log p_i - \delta_i \log r_i) = \sum_i \Bigg(& (k_1(p \circ p - p) + k_2((p \circ p) \circ p - p))_i (\log p_i - \log r_i) \\ & + \delta_i \frac{k_1(p \circ p - p)_i + k_2((p \circ p) \circ p - p)_i}{p_i} \\ & - \delta_i \frac{k_1(r \circ r - r)_i + k_2((r \circ r) \circ r - r)_i}{r_i} \Bigg) \\ = {} & k_1 I_2 + k_2 I_3 \end{aligned} \tag{11}$$

where

$$I_2 \equiv \sum_i \left((p \circ p - p)_i (\log p_i - \log r_i) + \delta_i \frac{(p \circ p - p)_i}{p_i} - \delta_i \frac{(r \circ r - r)_i}{r_i} \right) . \tag{12}$$

and

$$I_3 \equiv \sum_i \left(((p \circ p) \circ p - p)_i (\log p_i - \log r_i) + \delta_i \frac{((p \circ p) \circ p - p)_i}{p_i} - \delta_i \frac{((r \circ r) \circ r - r)_i}{r_i} \right) . \tag{13}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t} H = k_2 I_1 + k_1 I_2 + k_2 I_3 . \tag{14}$$

Since

$$\sum_i p_i\alpha_i = 0 \qquad \sum_i q_i\alpha_i = 0$$

and

$$\sum_i(\varepsilon_i\alpha_i + \delta_i\gamma_i) = \sum_{i,j}(a_{ij}\varepsilon_i\delta_j + a_{ij}\delta_i\varepsilon_j) = 0$$

we have

$$\begin{aligned}
I_2 &= \sum_i(p_i\alpha_i \log p_i + \delta_i\alpha_i - p_i\alpha_i \log r_i - \delta_i\gamma_i) \\
&= \sum_i(p_i\alpha_i \log(q_i + \delta_i) + \delta_i\alpha_i - p_i\alpha_i \log(q_i + \varepsilon_i) - \delta_i\gamma_i) \\
&\approx \sum_i\left(p_i\alpha_i\left(\frac{\delta_i}{q_i} - \frac{\varepsilon_i}{q_i}\right) + \delta_i\alpha_i - \delta_i\gamma_i\right) \\
&\approx \sum_i(2\delta_i\alpha_i - \varepsilon_i\alpha_i - \delta_i\gamma_i) \\
&= \sum_i(2\delta_i\alpha_i + 2q_i\alpha_i - 2q_i\alpha_i - \varepsilon_i\alpha_i - \delta_i\gamma_i) \\
&= \sum_i(2p_i\alpha_i - 2q_i\alpha_i - \varepsilon_i\alpha_i - \delta_i\gamma_i) = 0 .
\end{aligned} \tag{15}$$

Putting

$$A_i = \frac{((p\circ p)\circ p - p\circ p)_i}{p_i} \qquad \text{and} \qquad \Gamma_i = \frac{((r\circ r)\circ r - r\circ r)_i}{r_i}$$

we have (Itoh 1981, p 56)

$$\sum_i q_iA_i = 2\sum_i q_i\alpha_i^2 \qquad \text{and} \qquad \sum_i q_i\Gamma_i = 2\sum_i q_i\gamma_i^2 .$$

Taking into account $(p\circ p)\circ p - p = (p\circ p)\circ p - p\circ p + p\circ p - p$ and

$$I_2 = \sum_i(p_i\alpha_i \log p_i + \delta_i\alpha_i - p_i\alpha_i \log r_i - \delta_i\gamma_i) \approx 0$$

we have

$$\begin{aligned}
I_3 = \sum_i((p_iA_i + p_i\alpha_i)\log p_i + \delta_i(A_i + \alpha_i) - (p_iA_i + p_i\alpha_i)\log r_i - \delta_i(\Gamma_i + \gamma_i))& \\
\approx \sum_i(p_iA_i \log(q_i + \delta_i) + \delta_iA_i - p_iA_i \log(q_i + \varepsilon_i) - \delta_i\Gamma_i)& \\
\approx \sum_i\left(p_iA_i\left(\frac{\delta_i}{q_i} - \frac{\varepsilon_i}{q_i}\right) + \delta_iA_i - \delta_i\Gamma_i\right)& \\
= \sum_i\left((q_i + \delta_i)A_i\left(\frac{\delta_i}{q_i} - \frac{\varepsilon_i}{q_i}\right) + \delta_iA_i - \delta_i\Gamma_i\right)& \\
\approx \sum_i(2\delta_iA_i - \varepsilon_iA_i - \delta_i\Gamma_i) .&
\end{aligned} \tag{16}$$

Since $\delta_i + q_i = p_i$ and $\sum_i p_i A_i = 0$,

$$\begin{aligned}\sum_i (2\delta_i A_i - \varepsilon_i A_i - \delta_i \Gamma_i) &= \sum_i (2\delta_i A + 2q_i A_i - 2q_i A_i - \varepsilon_i A_i - \delta_i \Gamma_i)\\ &= \sum_i (-\varepsilon_i A_i - \delta_i \Gamma_i - 2q_i A_i)\\ &= -\sum_i (\varepsilon_i A_i + \delta_i \Gamma_i) - 4\sum_i q_i \alpha_i^2 .\end{aligned} \tag{17}$$

Since

$$\begin{aligned}(p \circ p - p) \circ p &= 2\left(\sum_{i,j=1}^{m} a_{ij} p_i p_j E_i\right) \circ \left(\sum_{k=1}^{m} p_k E_k\right)\\ &= \sum_{i,j,k=1}^{m} a_{ij} p_i p_j p_k (E_i + 2a_{ik} E_i + E_k + 2a_{ki} E_k)\end{aligned} \tag{18}$$

we have

$$\sum_\rho \varepsilon_\rho A_\rho = \sum_{\rho,j,k} \varepsilon_\rho a_{\rho j} p_j p_k + 2\sum_{\rho,j,k} \varepsilon_\rho a_{\rho j} a_{\rho k} p_j p_k + \sum_{i,j,\rho} \varepsilon_\rho a_{ij} p_i p_j + 2\sum_{i,j,\rho} \varepsilon_\rho a_{ij} a_{\rho i} p_i p_j . \tag{19}$$

Since

$$\sum_j a_{ij} \varepsilon_j = \sum_j a_{ij} r_j \qquad \text{and} \qquad \sum_j a_{ij} \delta_j = \sum_j a_{ij} p_j$$

we have

$$\begin{aligned}\sum_{\rho,j,k} \varepsilon_\rho a_{\rho j} p_j p_k &= -\sum p_j \gamma_j\\ \sum_{\rho,j,k} \varepsilon_\rho a_{\rho j} a_{\rho k} p_j p_k &= \sum_\rho \varepsilon_\rho \alpha_\rho^2\\ \sum_{i,j,\rho} \varepsilon_\rho a_{ij} a_{\rho i} p_i p_j &= -\sum_i p_i \alpha_i \gamma_i .\end{aligned}$$

Hence

$$\sum_\rho \varepsilon_\rho A_\rho = -\sum_j p_j \gamma_j + 2\sum_\rho \varepsilon_\rho \alpha_\rho^2 - 2\sum_i p_i \alpha_i \gamma_i \tag{20}$$

$$\sum_\rho \delta_\rho \Gamma_\rho = -\sum_j r_j \alpha_j + 2\sum_\rho \delta_\rho \gamma_\rho^2 - 2\sum_i r_i \alpha_i \gamma_i . \tag{21}$$

The middle summations on the right-hand side of (20) and (21) will be neglected because these terms are negligible near equilibrium.

Since

$$\sum_i (r_i \alpha_i + p_i \gamma_i) = \sum_{i,j} (a_{ij} r_i p_j + a_{ij} p_i r_j) = 0$$

$$\sum_i p_i \alpha_i \gamma_i \approx \sum_i q_i \alpha_i \gamma_i \qquad \text{and} \qquad \sum_i r_i \alpha_i \gamma_i \approx \sum_i q_i \alpha_i \gamma_i$$

we have

$$I_3 \approx -4\sum_i q_i \alpha_i^2 + 4\sum_i q_i \alpha_i \gamma_i . \tag{22}$$

Thus we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_i p_i \log\frac{p_i}{r_i} = k_2 I_1 + k_1 I_2 + k_2 I_3 \approx -2k_2\sum_i q_i(\alpha_i - \gamma_i)^2 . \tag{23}$$

□

Example 1. We consider the model which satisfies (i) and (iv) of section 2, and the following (ii′) and (iii′).

(ii′) Each particle participates in triple collision on average dt times per time length dt. A triple collision is expressed as in figure 1, in which particle X collides with particle Y, and Y collides with particle Z.

(iii′) Each colliding triple is equally likely to be chosen.

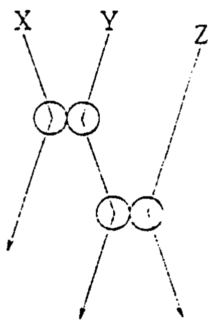

Figure 1. Two successive binary collisions make a ternary collision in which particle X collides with particle Y, and Y collides with particle Z.

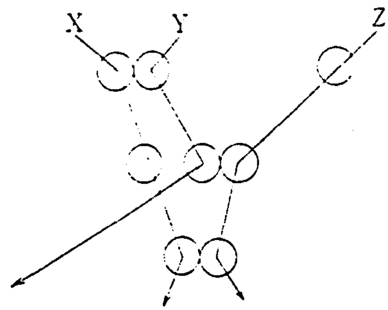

Figure 2. Three successive binary collisions make a ternary collision in which particle X collides with particle Y, Y collides with particle Z, and particle Z collides with particle X.

From the above setting, we have the following equation with $p(t) = \sum_{i=1}^{m} p_i(t)E_i \in A^m$:

$$\frac{\mathrm{d}}{\mathrm{d}t}p(t) = \tfrac{1}{3}p(t)\circ p(t) + \tfrac{2}{3}p(t)\circ(p(t)\circ p(t)) - p(t)$$

Each of $n\,\mathrm{d}t$ particles participates in a ternary collision in time interval dt. Each $\frac{1}{3}n\,\mathrm{d}t$ particles of them takes the part of X in figure 1. Each of the remaining $\frac{2}{3}n\,\mathrm{d}t$ particles takes the part of Y or Z in figure 1. So the above equation is reasonable.

In the case of low density, we need not consider the effect of triple collisions. So k_2 is very small. In the case of higher density k_2 is not so small.

Example 2. In this example, (ii′) of the previous example is replaced by the following (ii″):

(ii″) Each particle participates in a triple collision on average dt times per time length dt. A triple collision consists of three successive binary collisions as in figure 2, that is, particle X collides with particle Y, Y collides with particle Z, and finally particle Z collides with particle X.

We consider $p \in A^3$ where $a_{12} = a_{23} = a_{31} = \frac{1}{2}$, in which case $q = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Then the law of change is given by $(\mathrm{d}/\mathrm{d}t)p = p\circ(p\circ p) - p$.

Since $p \circ (p \circ p) - p = p \circ (p \circ p) - (\sum_{i=1}^{3} p_i)^2 p$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} p_1 = -p_1 p_2^2 - 2p_1^2 p_2 + p_3 p_1^2 + 2p_3^2 p_1$$
$$\frac{\mathrm{d}}{\mathrm{d}t} p_2 = -p_2 p_3^2 - 2p_2^2 p_3 + p_1 p_2^2 + 2p_1^2 p_2$$
$$\frac{\mathrm{d}}{\mathrm{d}t} p_3 = -p_3 p_1^2 - 2p_3^2 p_1 + p_2 p_3^2 + 2p_2^2 p_3$$

where $-p_1 p_2^2$ corresponds to the event that one particle of species 1 interacts with two particles of species 2 and changes to one particle of species 2, $-2p_1^2 p_2$ corresponds to the event that two particles of species 1 interact with one particle of species 2 and change to two particles of species 2, $p_3 p_1^2$ corresponds to the event that one particle of species 3 interacts with two particles of species 1 and changes to one particle of species 1, $2p_3^2 p_1$ corresponds to the event that two particles of species 3 interact with one particle of species 1 and change to two particles of species 1.

A triple which consists of one particle of species 1, one particle of species 2, and one particle of species 3, makes no change for p_1 in total. Thus we see that

$$\frac{\mathrm{d}}{\mathrm{d}t} p_1 = -p_1 p_2^2 - 2p_1^2 p_2 + p_3 p_1^2 + 2p_3^2 p_1$$

is reasonable.

4. Discussion

For conservative linear systems (finite-state Markov processes in discrete or continuous time), the relative entropy of two distinct trajectories is a monotonically decreasing function of time. The two distinct trajectories of our nonlinear conservative system also display monotonically decreasing relative entropy near equilibrium.

For Lotka–Volterra systems of binary interactions with anti-symmetry, the relative entropy of two distinct trajectories continues to oscillate under the motion. If a Lotka–Volterra system has ternary interactions with anti-symmetry as well as binary interaction, the relative entropy of two distinct trajectories has damped oscillations with time far from equilibrium, and is monotonically decreasing near equilibrium as can be observed in the numercal study (figure 3).

To understand the values k_1 and k_2 of (4), we give a discrete model of the binary and the ternary interaction to simplify the discussion. Consider an occupancy problem for a system of n particles (Johnson *et al* 1992, pp 420–2). Suppose there are c places, $1, 2, \ldots, c$, in which each particle can be. In unit time the n particles are distributed on the c places at random. All c^n arrangements are assumed to be equally probable. Two particles, in a particular place, are considered to be in a binary collision. The three particles, in a particular place, are considered to be in a ternary collision given in figure 2. The probability $\Pr(X = x)$ that there are x particles ($x \leqslant n$) in a particular place is $\Pr(X = x) = {}_nC_x(1/c)^x(1 - 1/c)^{n-x}$. The value $\Pr(X = 3)/\Pr(X = 2)$ could represent k_2/k_1. Neglecting collisions of order higher than three, we have equation (4). When $n = c^{9/10}$ for $c = 100\,000$, $\Pr(X = 3)/\Pr(X = 2)$ is approximately 0.105. $\Pr(X = 4)/\Pr(X = 2)$ is approximately 0.0083. In figure 3, we give a numerical study for the case $k_1 = 20$ and $k_2 = 2.1$, neglecting interactions higher than three for the two trajectories which start from $(0.3, 0.3, 0.4)$ and $(0.35, 0.35, 0.3)$ for $p \in A^3$ where $a_{12} = a_{23} = a_{31} = \frac{1}{2}$. Our numerical studies show that k_2 seems to determine the speed of approach to equilibrium almost independently of k_1.

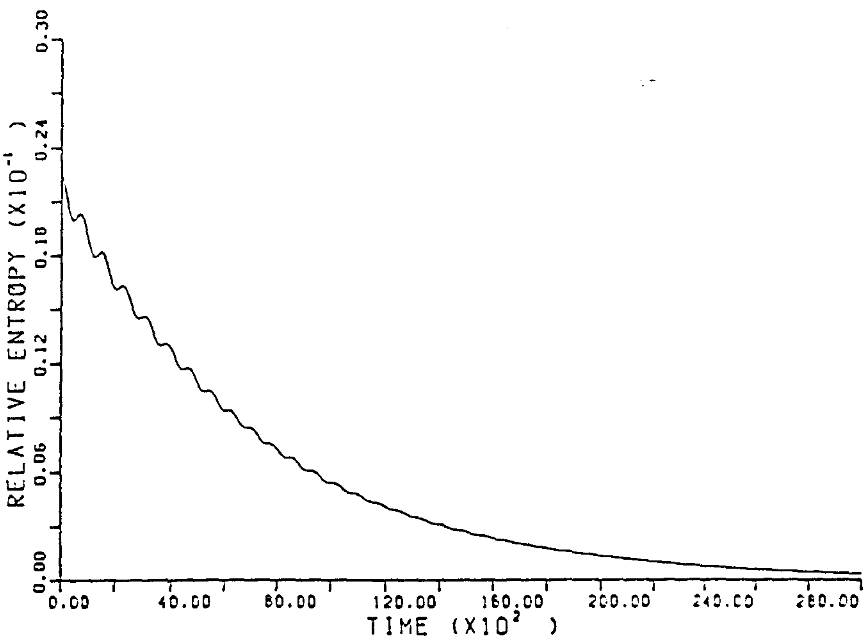

Figure 3. The relative entropy of the two distinct trajectories has damped oscillations with time far from equilibrium and is monotonically decreasing near equilibrium. The two trajectories start from (0.3, 0.3, 0.4) and (0.35, 0.35, 0.3) for $p \in \Delta^3$ with $a_{12} = a_{23} = a_{31} = \frac{1}{2}$, where $k_1 = 20$ and $k_2 = 2.1$.

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