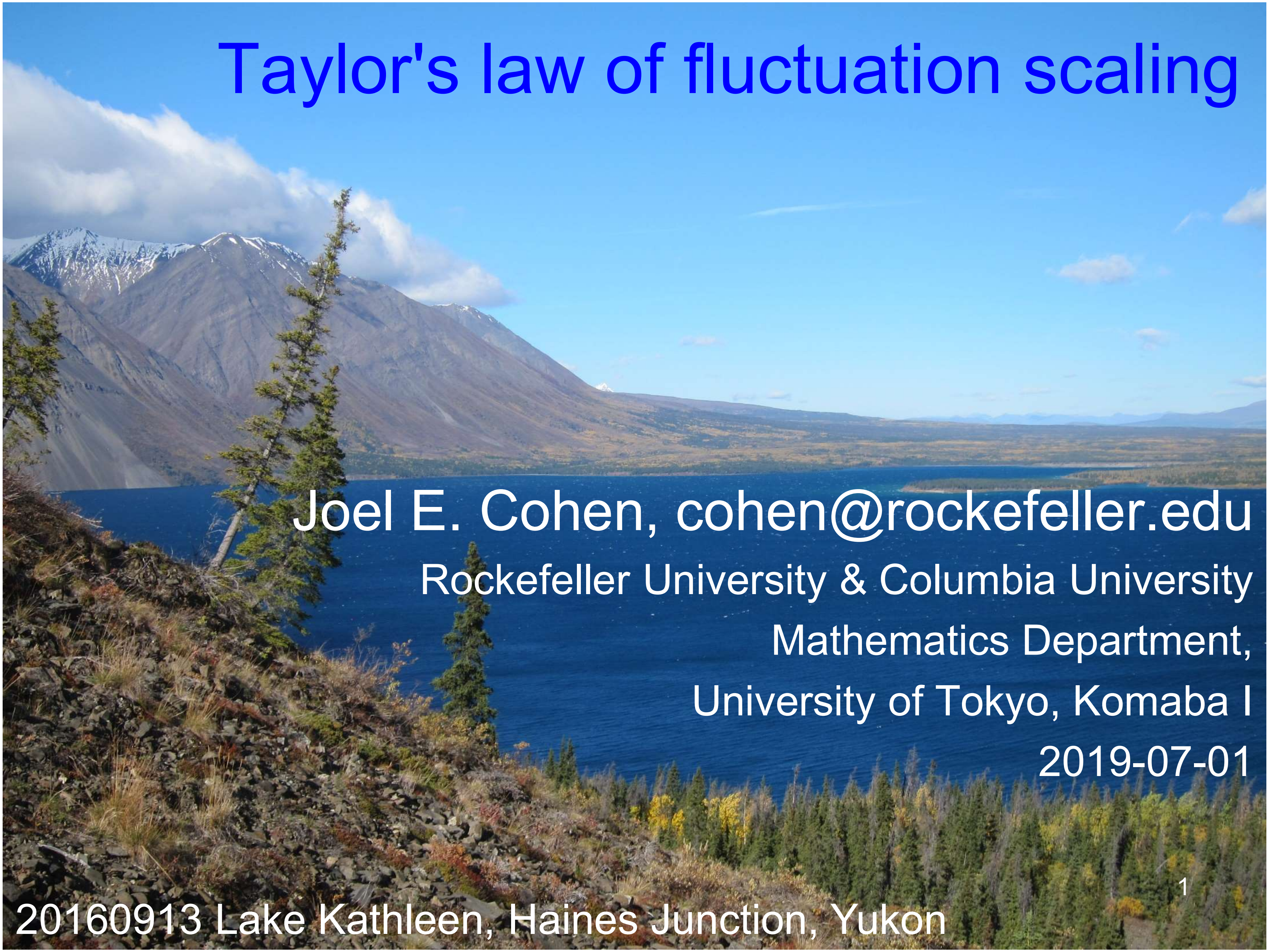

Taylor's law of fluctuation scaling

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2019-07-01

20160913 Lake Kathleen, Haines Junction, Yukon

Outline

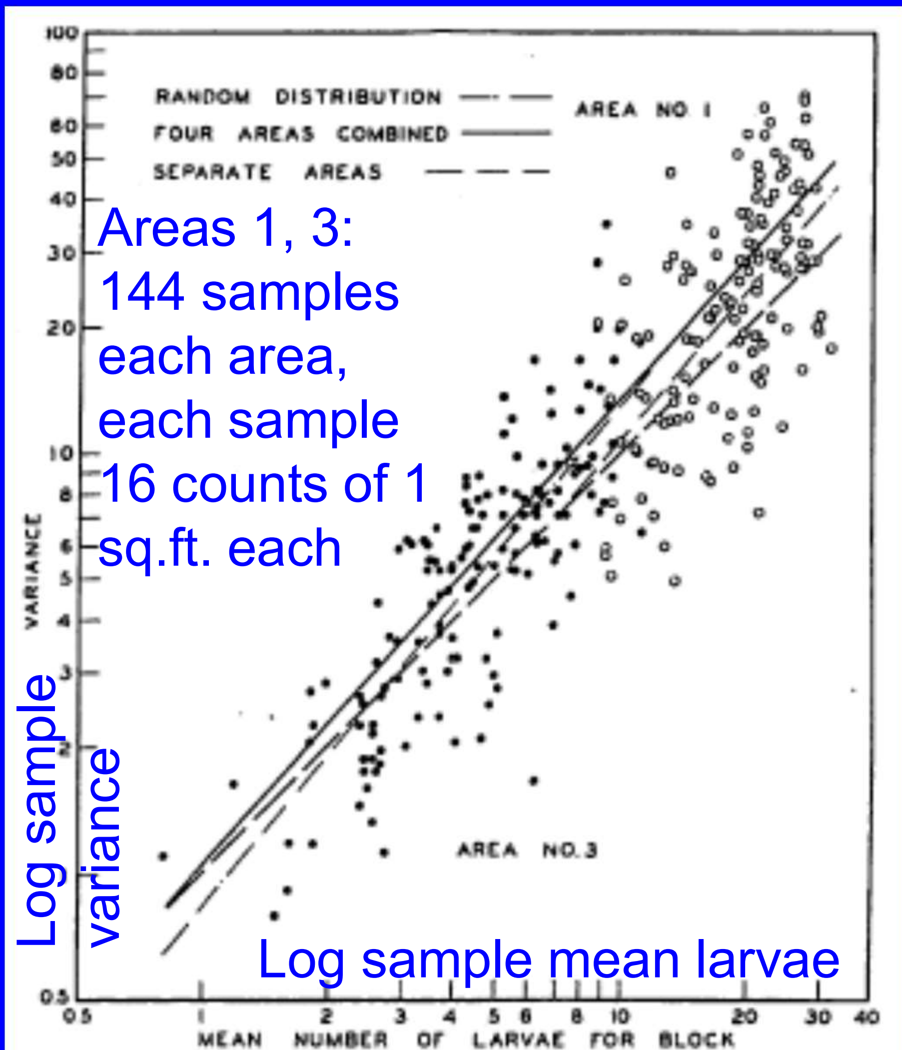
1. ➔ What is Taylor's law (TL)?
2. Empirical examples from my work
3. TL does not always hold
4. Theories of TL
5. Conclusions

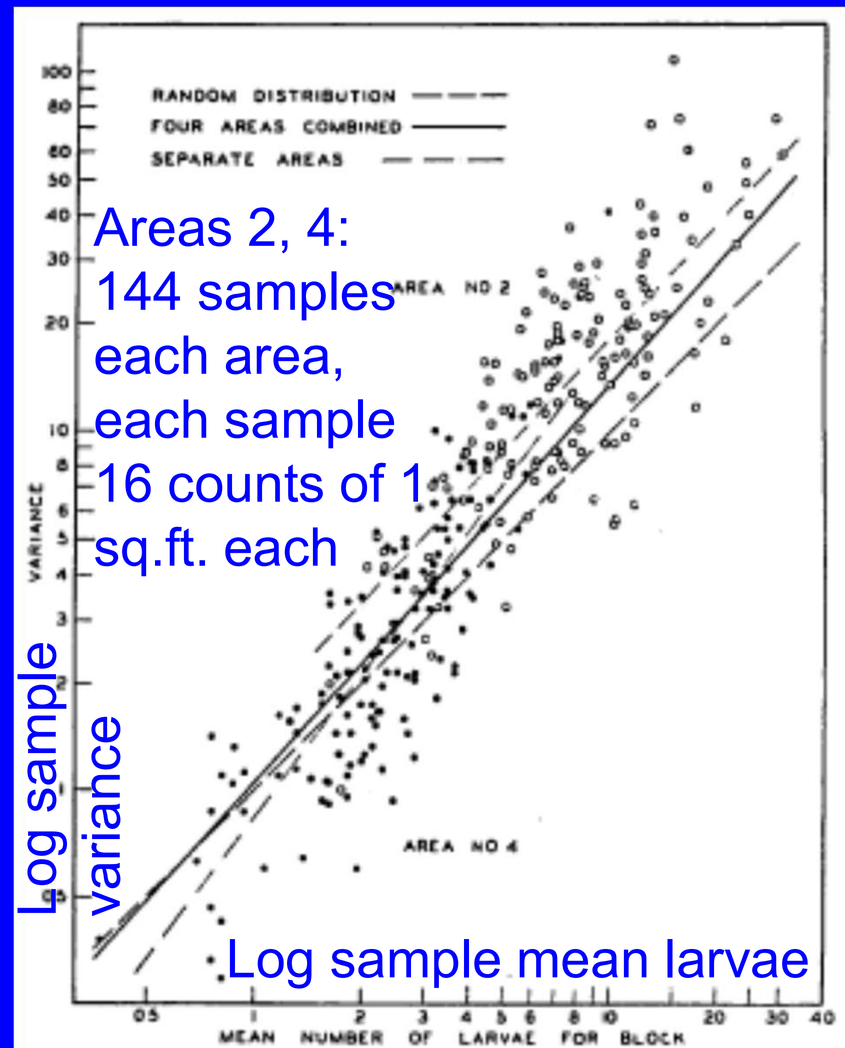
TL data structure: multiple samples, each with multiple observations

Sample number →	*j*=1	*j*=2	*j*=3	*j*=…
Population size or density in units (quadrats, plots, transects, counties, states, years, days)	x_{11}	x_{12}	x_{13}	$x_{...}$
	x_{21}	x_{22}	x_{23}	…
	x_{31}	x_{32}	x_{33}	…
		x_{42}	x_{43}	…
		x_{52}		…
Mean (weighted)	m_1	m_2	m_3	m…
Variance (weighted)	v_1	v_2	v_3	v…

Japanese beetle larvae $v_j = a m_j^b$

Chester I. Bliss *J. of Economic Entomology* 1941

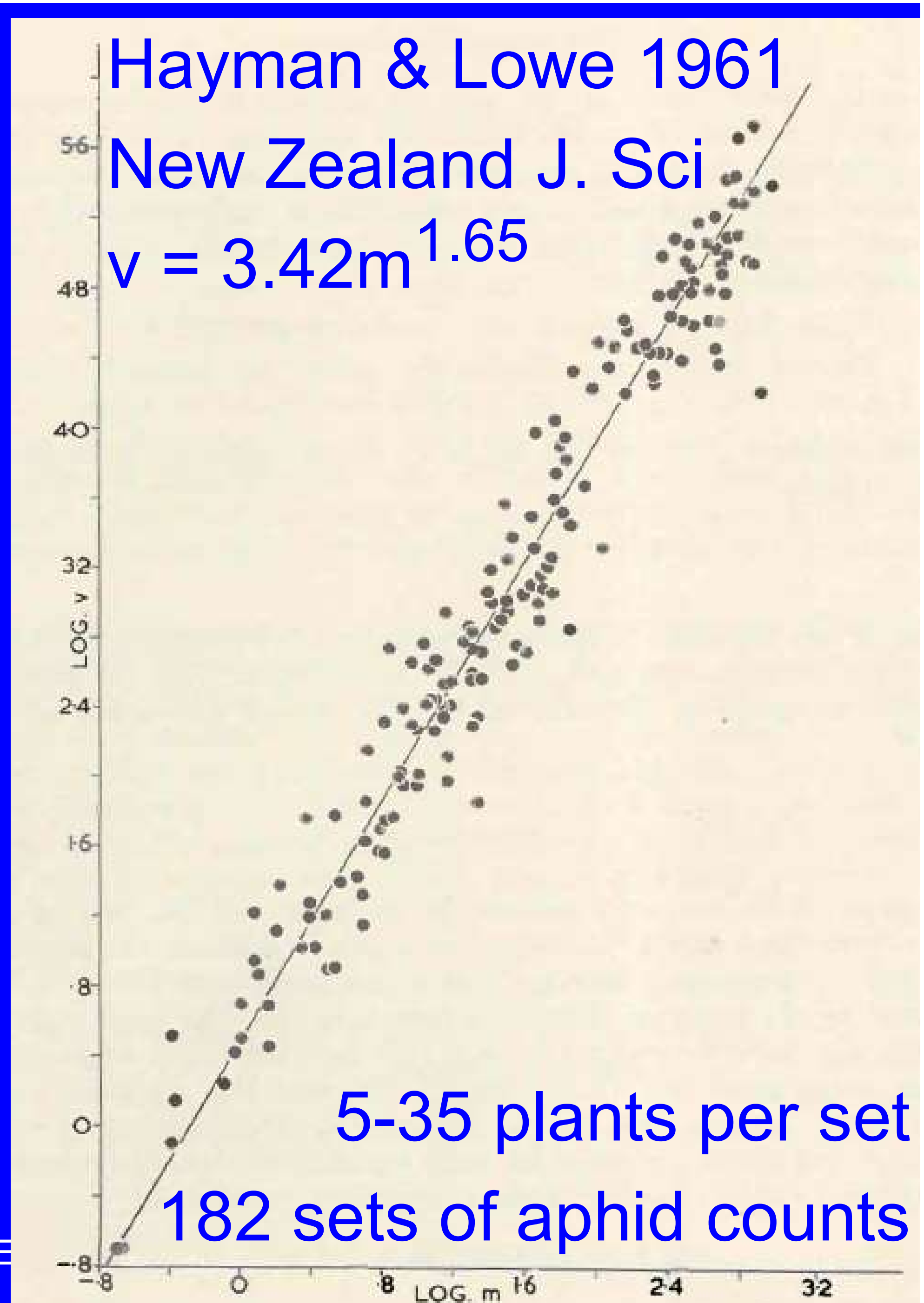




Count aphids on each plant in a set of 5-35 plants. Count 182 sets of plants.

Both sample variance v & sample mean m have sampling uncertainty, but log v has about 6x the error variance of log m.

Goal of study was to stabilize variance for ANOVA.



Taylor's law *Nature* 1961

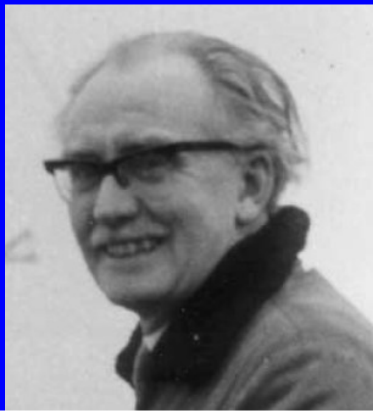
In multiple sets of samples, the variance of population density is proportional to a power of the mean population density.

$$\text{variance} = a(\text{mean})^b, a > 0.$$

$$\log(\text{variance}) = \log(a) + b \cdot \log(\text{mean}).$$

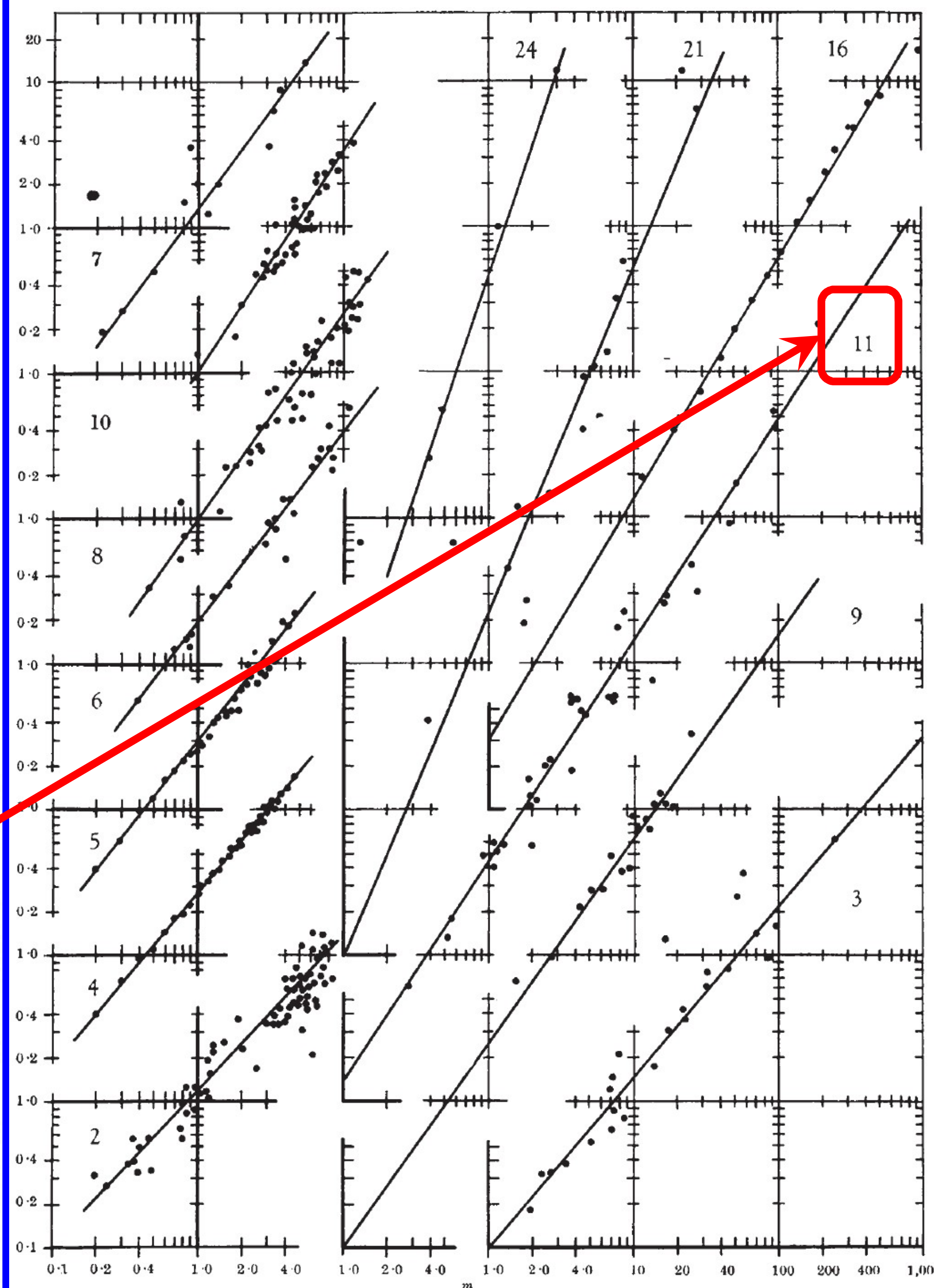
$$\text{variance}/(\text{mean})^b = a, \quad a > 0.$$

Taylor stated no model of error (deviations from exact equality).



Lionel Roy Taylor
(1924–2007)

Name	Site and sample
1 Shellfish on seashore, *Tellina tenuis* da Costa, Eulamellibranchiata : Mollusca	Sand, 63 units, various sizes
2 European chafer larvæ, *Amphimallon majalis* Raz. (= *Melolontha melolontha* L.), Coleoptera : Insecta	Pasture soil, 25 units, each 1 ft. sq.
3 Flying insects, various orders : Insecta	Open air, 16–104 units aerial density
4 Wireworms, *Agriotes* spp. mainly *obscurus*, Coleoptera : Insecta	Grassland soil 20 units, 4 in. cores
5 Wireworms, *Agriotes* as above	Arable land soil 20 units, 4 in. cores
6 Wireworms, *Limonius* spp., Coleoptera : Insecta	Arable land soil, 175 units, each 1 ft. sq.
7 Gall midge larvæ, *Jaapiella medicaginis* (Rub.), Diptera : Insecta	Lucerne field soil, 10 units, 4 in. cores
8 Spruce budworm larvæ, *Choristoneura fumifera* (Clem.), Lepidoptera : Insecta	Fir foliage, 25 units, larvæ/twig
9 Virus lesions, tobacco necrosis virus	Bean leaves, 4 units, lesions/half leaf
10 Colorado beetle adults, *Leptinotarsa decemlineata* Say., Coleoptera : Insecta	Potato foliage, 2,304 counts : insects/2 ft. row
11 Japanese beetle larvæ, *Popillia japonica* New., Coleoptera : Insecta	Soil, 10,000 units each 1 ft. sq.*
12 Macro-zooplankton	Water, net collection, slide count, 10 areas
13 Macro-zooplankton	As above, 50 areas
14 Ticks, *Ixodes ricinus* L., Acarina : Arachnida	Sheep, 20–86 units, ticks/sheep
15 Enchytraeid worms, mainly *Fridericia bisetosa* (Lev.), Enchytraeidae : Annelida	Pasture, 60–150 units, 3·6 cm. cores
16 Corn borer larvæ, *Pyrausta nubilalis* (Hubn.), Lepidoptera : Insecta	Maize stalks, 2 stages†
17 Thrips, *Thrips imaginis* (Bagnall), Thysanoptera : Insecta	Rose flowers, 20 units, thrips/rose
18 Leather-jackets, *Tipula* spp., Diptera : Insecta	Soil, 2 units, Nos./sq. ft.
19 Earthworms, all stages, *Allolobophora chlorotica* (Sav.), Oligochaeta : Annelida	Grassland, 4 units, 18 in. sq.
20 Red spider mite, eggs and adults, *Metatetranychus ulmi* (Koch), Acarina : Arachnida	Apple leaves, 20 units, mites/leaf
21 Haddock, *Melanogrammus aeglefinus*, Gadidae : Pisces	Sea, 4–47 units, Nos./trawl
22 Earthworms, all stages, *Allolobophora caliginosa* (Sav.), Oligochaeta : Annelida	Grassland, 4 units, 18 in. sq.
23 Symphyla, *Symphylella* spp., Symphyla : Myriapoda	Various soils, 60–120 units, 2½ in. cores
24 Symphyla, *Scutigerella* spp., Symphyla : Myriapoda	Various soils, 60–140 units, 2½ in. cores



TL matters practically because variability is fundamental.

- Fluctuations of forests, fisheries, infectious diseases, disease vectors, tornados
- Conservation of endangered species
- Sampling insect pests of cotton & soybeans, fishery stocks
- Linking levels of biological organization: variance-body mass allometry
- Evaluation of human population projections

Taylor's law(s) mathematically

Given an index set $S \neq \emptyset$ with element(s) s, $\{X(s)\}_{s \in S}$ is a family of nonnegative random variables with finite, positive first 2 moments.

Population TL: ∃ real constants $a > 0, b$, such that $\forall s \in S$, $\quad Var(X(s)) = a(E(X(s)))^b$.

Sample of size $n > 1$ is a set $\{X_1(s), \dots, X_n(s)\}$ of n iid copies of $X(s)$, with sample mean $m_n := (X_1 + \cdots + X_n)/n$, sample variance s_n^2.

Sample TL: $\log s_n^2 \approx \log a + b \log m_n$ or

$$\frac{s_n^2}{(m_n)^b} \text{"is close to" } a > 0.$$

Slope *b* in TL is "elasticity."

Taylor's law says: variance ≈ *a*(mean)^{b}.

Then $b \approx \dfrac{d \log variance}{d \log mean} = \dfrac{\frac{1}{var} \times d\, var}{\frac{1}{mean} \times d\, mean}$

≈ % change in variance for 1% change in mean.

b = "elasticity of variance with respect to mean" (in economists' use of "elasticity").

b = 2 iff coefficient of variation (SD/mean) & signal-to-noise ratio (mean/SD) are constant (regardless of the value of the mean).

Slope *b* in TL is independent of scale of measurement.

If $s^2 = am^b$ for r.v. X, & $Y = kX, k > 0$, then mean of Y is $\mu = km$, variance of Y is

$$\sigma^2 = k^2 s^2 = k^2 a m^b = k^2 a \left(\frac{\mu}{k}\right)^b = k^{2-b} a \mu^b.$$

Y obeys TL power law with same exponent *b*, coefficient ak^{2-b}.

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Spatial & temporal Taylor's laws

Let $X(s, t)$ = population density at *s* in space & at *t* in time.

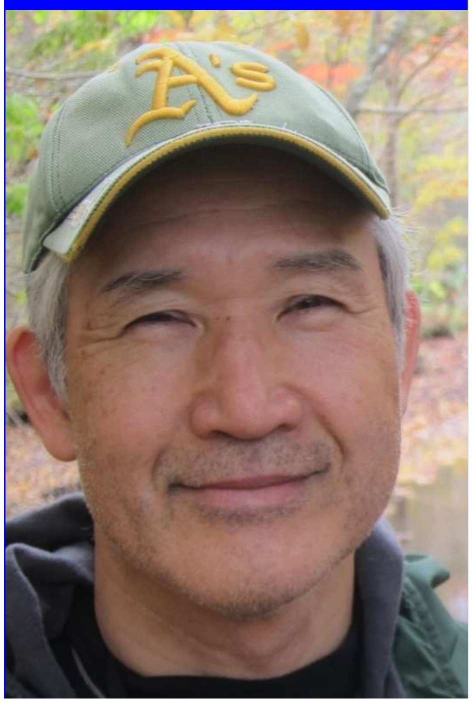
Spatial TL: for each t, calculate mean & variance averaging over space *s.*

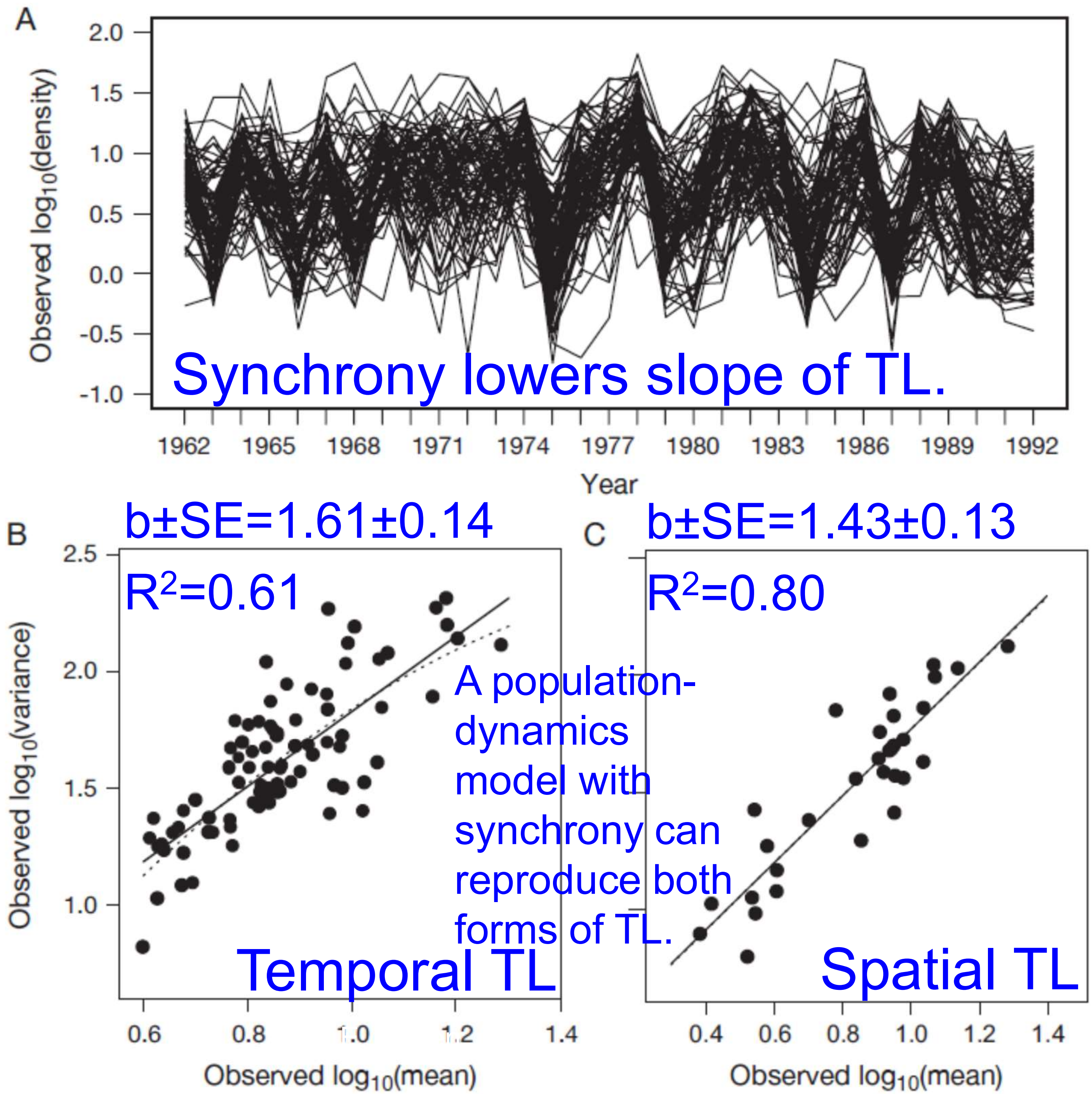
Temporal TL: for each s, calculate mean & variance averaging over time *t.*

Voles

85 populations, 31 years Hokkaido, Japan

Cohen & Saitoh *Ecology* 2016





Questions re spatial & temporal TL

Let $X(s, t)$ = population density at *s* in space & at *t* in time.

Under what conditions on $X(s, t)$ do a spatial & temporal TLs hold simultaneously?

When spatial TL & temporal TL both hold, how are *b*(spatial) & *b*(temporal) related?

How should TL take account of spatial & temporal dependence?

Are TL & power spectrum related? If so, how?

Norway

Meng Xu, Pace U.

Helge Brunborg
Statistics Norway

Norway has 430 municipalities, 19 first-level administrative counties (*fylker*), & 5 regions, with complete counts from a population register from 1978 to 2010.

Norway annual registration

Observe $N_p(t)$ = number of people per square kilometer in county $p = 1, \ldots, 19$ in year $t = 1978, \ldots, 2010$.

county	1978	…	2010
1	$N_1(1978)$	…	$N_1(2010)$
…	…	…	…
19	$N_{19}(1978)$	…	$N_{19}(2010)$

Likewise for 430 municipalities, 5 regions.

Weighted population density

Let spatial unit s have area A_s, population size N_s, & density $D_s \coloneqq N_s / A_s$.

Weighted spatial mean m & weighted spatial variance s^2 of population density

$$m \coloneqq \frac{\sum_s w_s D_s}{\sum_s w_s}, s^2 \coloneqq \frac{\sum_s w_s (D_s - m)^2}{\sum_s w_s}$$

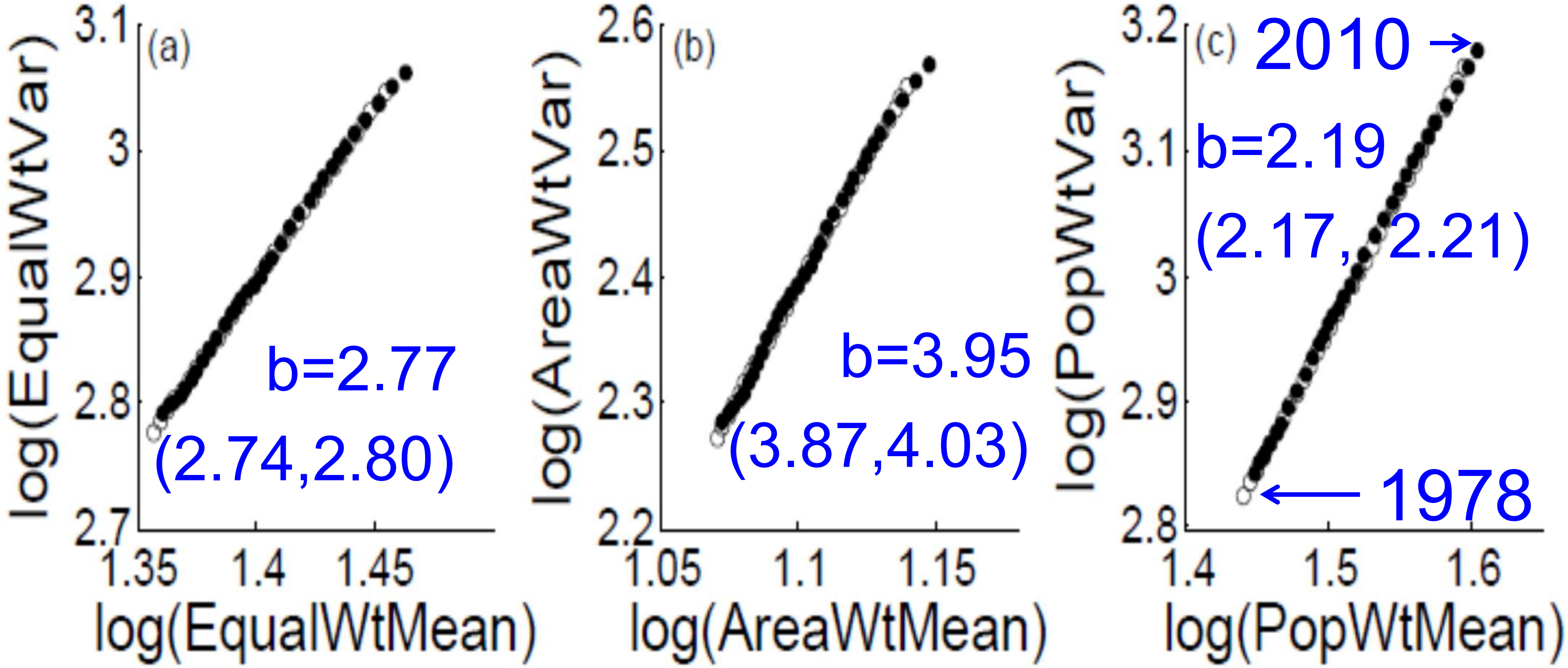
use 3 kinds of weights:

Equal $w_s = 1$, Areal $w_s = A_s$,

Population $w_s = N_s$.

Human population density of 18 counties of Norway (excluding Oslo) obeys spatial TL.

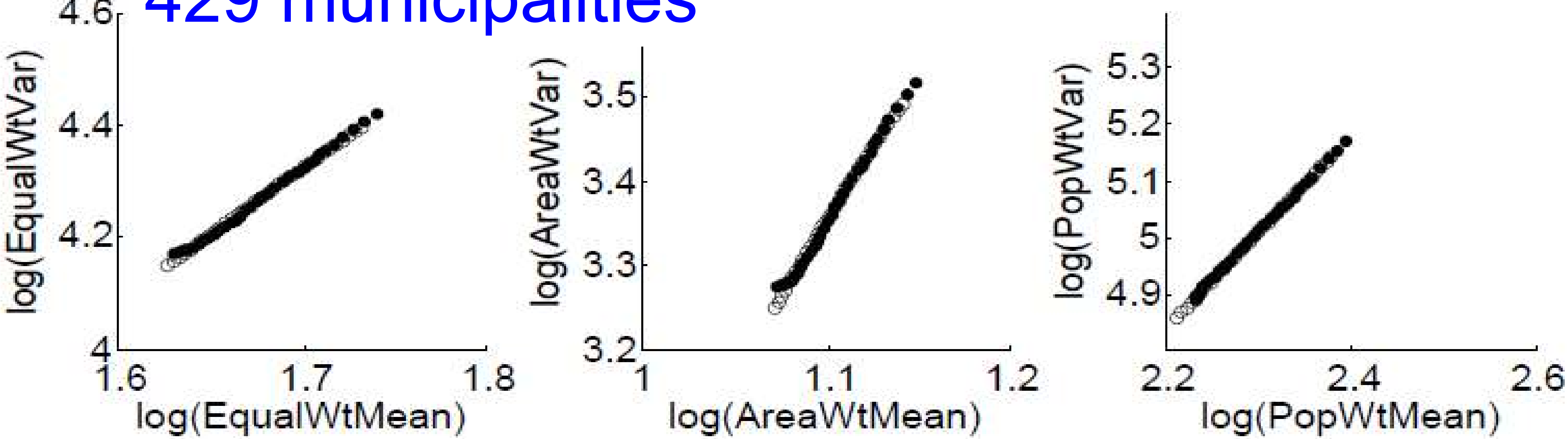
Cohen, Xu, Brunborg, *GENUS* 2013



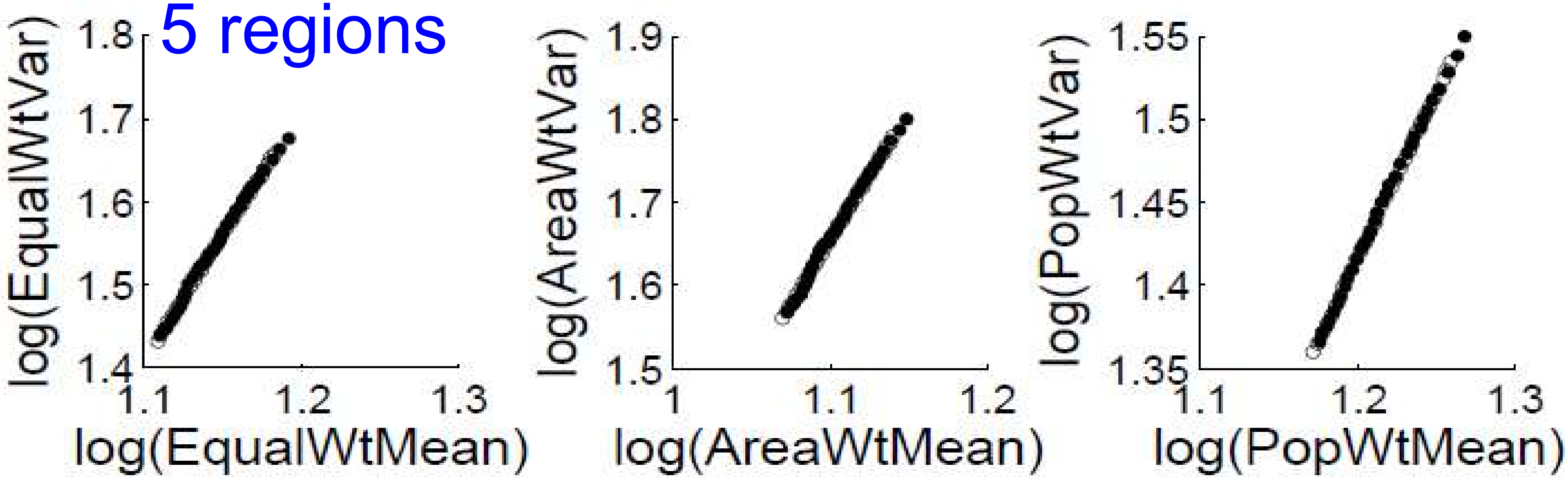
Norway excluding Oslo, 1978-2010

Cohen, Xu, Brunborg *GENUS* 2013

429 municipalities



5 regions



Questions re weighting & scales

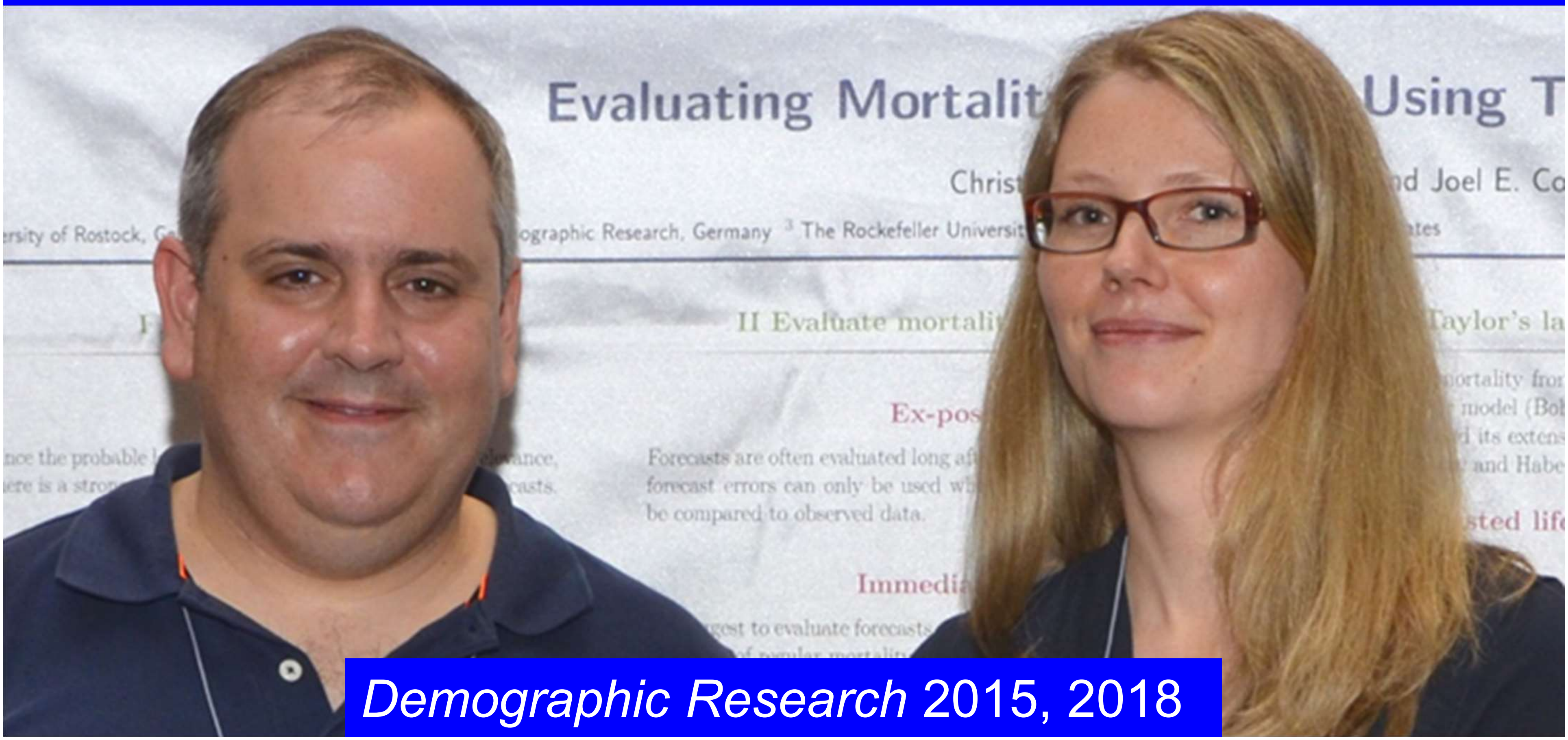
When TL is fitted using different weights (e.g., equal, areal, population weights), how are different estimates of *a* & *b* related?

When does TL hold simultaneously at different spatial scales, different temporal scales, & simultaneously at different spatial & temporal scales?

When TL holds at different scales, how are its parameters at different scales related?

How do human death rates vary over time, at each age, for each sex?

Roland Rau   Christina Bohk-Ewald



Demographic Research 2015, 2018

Mean & variance of mortality

Survival curve (life table) of life length X_t at time t is $\ell_{x,t} = \Pr\{X_t > x\}$.

Mortality μ at age x in year t is defined as

$$\mu_{x,t} = -\frac{1}{\ell_{x,t}}\left(\frac{d\ell_{x,t}}{dx}\right).$$

Temporal mean & temporal variance of mortality at age x are

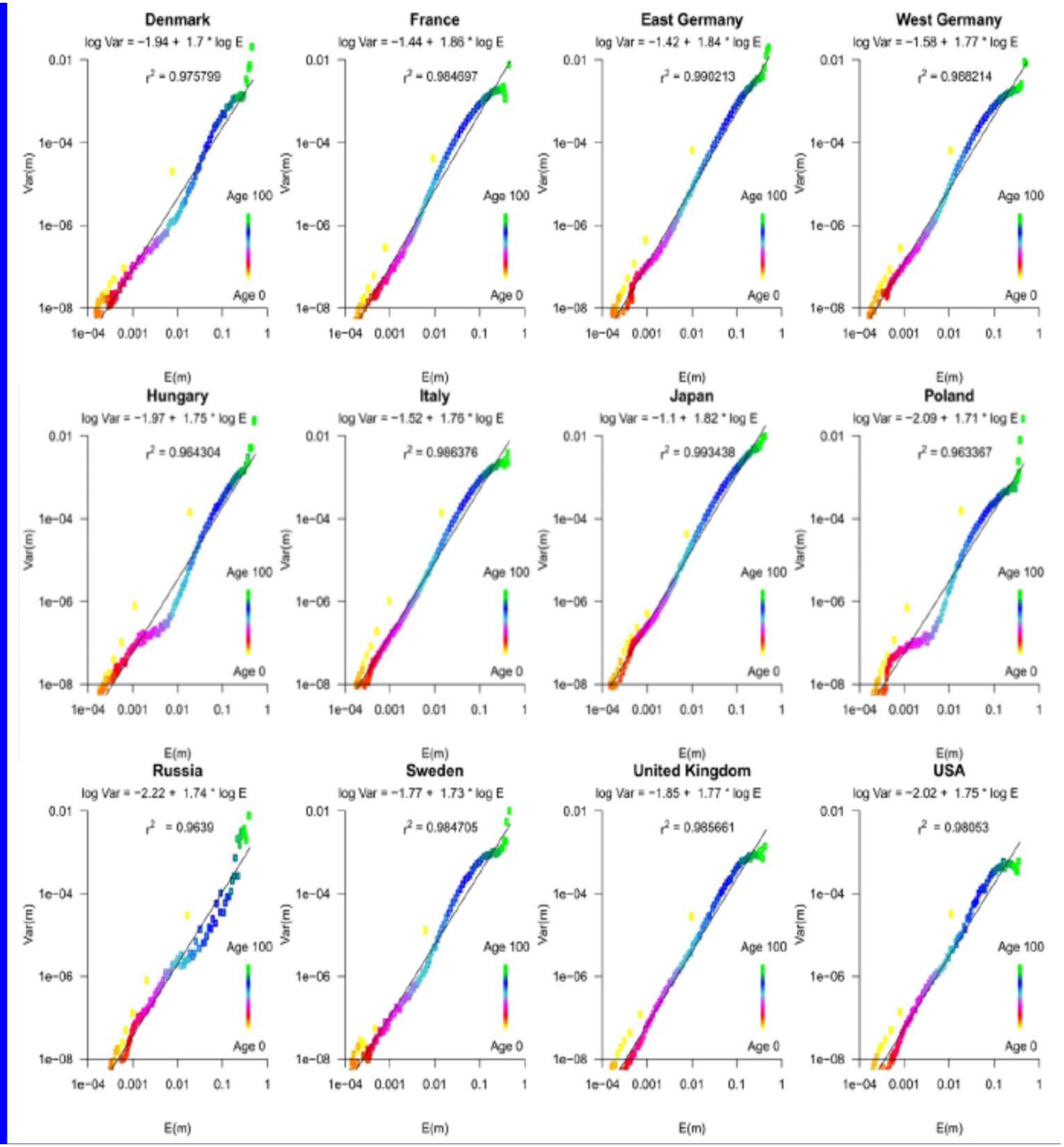
$$E(\mu_x) = \frac{1}{T}\sum_{t=1}^{T} \mu_{x,t}, \quad Var(\mu_x) = \frac{1}{T}\sum_{t=1}^{T}\left(\mu_{x,t} - E(\mu_x)\right)^2.$$

Human ♀ mortality: temporal TL

At each age 0-100 years, mean, var of age-specific ♀ death rate over 1960-2009

Cohen, Bohk-Ewald, Rau, *Demog. Res.* 2018

2019-07-11

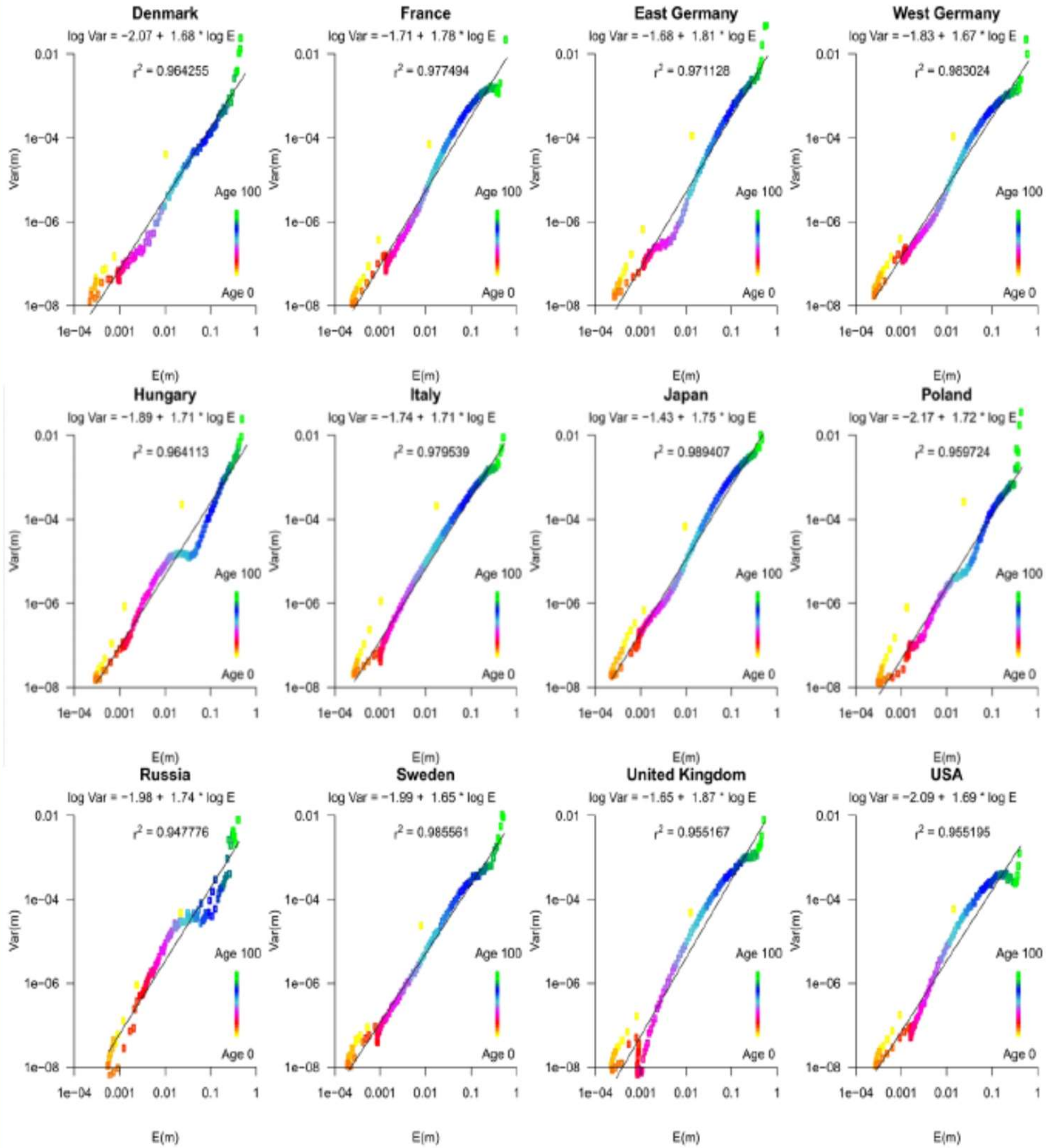


Human ♂ mortality: temporal TL

At each age 0-100 years, mean, var of age-specific ♂ death rate over 1960-2009

Cohen, Bohk-Ewald, Rau, *Demog. Res.* 2018

2019-07-11



Theory of TL in human mortality

Gompertz model of mortality *μ* at age *x* in year *t*:

$$\mu_{x,t} = \beta_t e^{\beta_t (x - M_t)}, \beta_t > 0, M_t > 0, \text{for } t = 1, \ldots, T, x = 0, \ldots, X.$$

$$E(\mu_x) = \frac{1}{T} \sum_{t=1}^{T} \mu_{x,t},$$

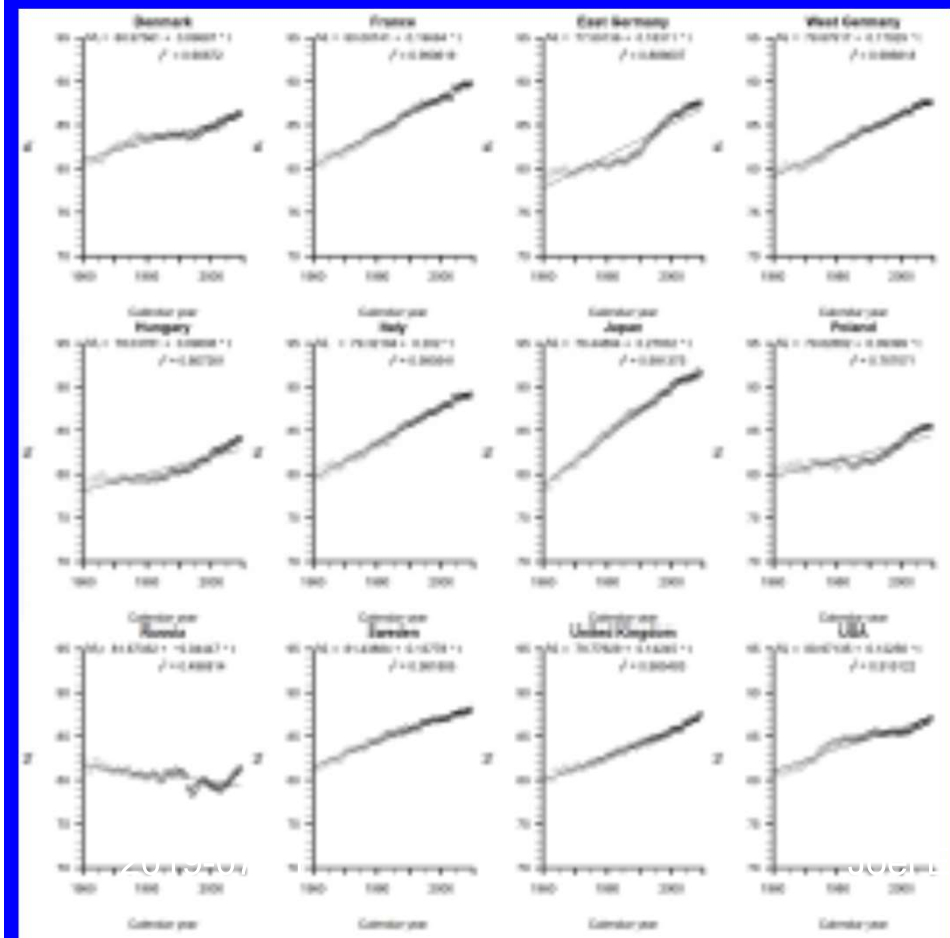
$$Var(\mu_x) = \frac{1}{T} \sum_{t=1}^{T} \left(\mu_{x,t} - E(\mu_x) \right)^2.$$

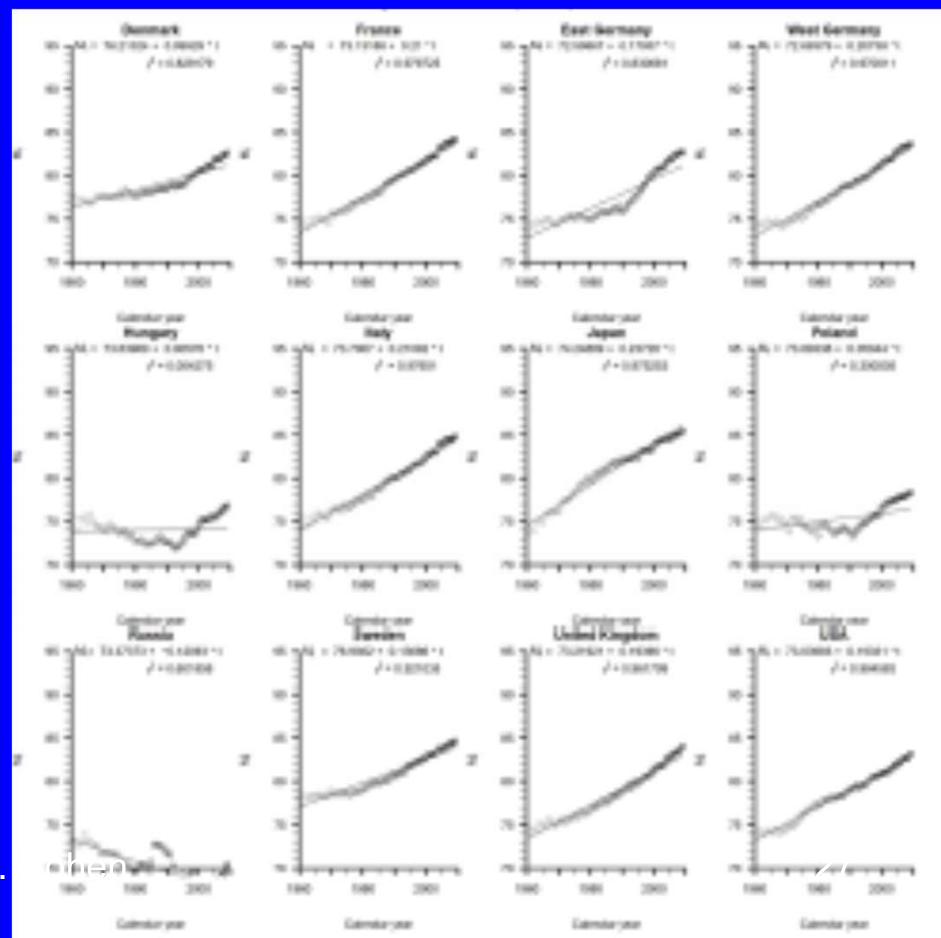
If modal age at death $M_t = v + wt > 0$, $v > 0, w \neq 0, t = 1, \ldots, T, \beta_t = \beta > 0$, then TL holds exactly with *b* = 2:

$$\log Var(\mu_x) = \log\left(\frac{K_2 - K_1^2}{K_1^2}\right) + 2 \cdot \log E(\mu_x), \text{ for } x = 1, \ldots, X.$$

Modal age at death grew nearly linearly.

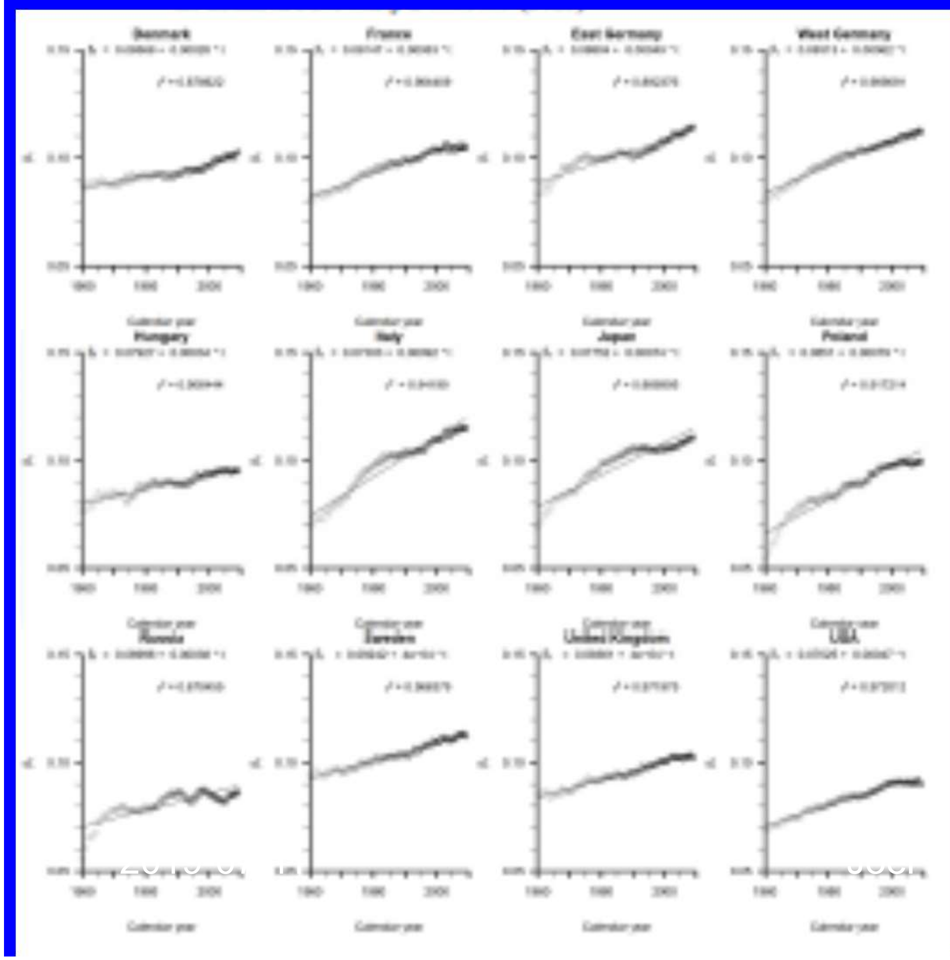
Female M_t 1960-2009 Male M_t 1960-2009



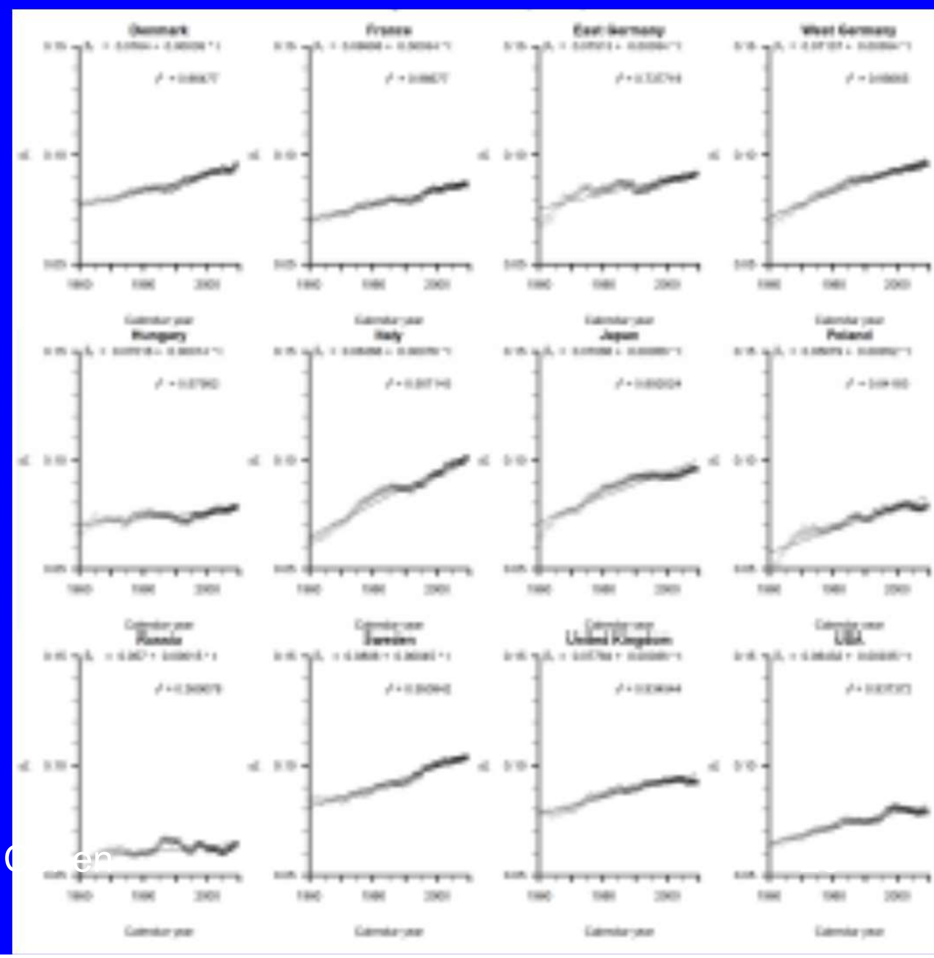


Estimated $b < 2$ almost always. β was not constant.

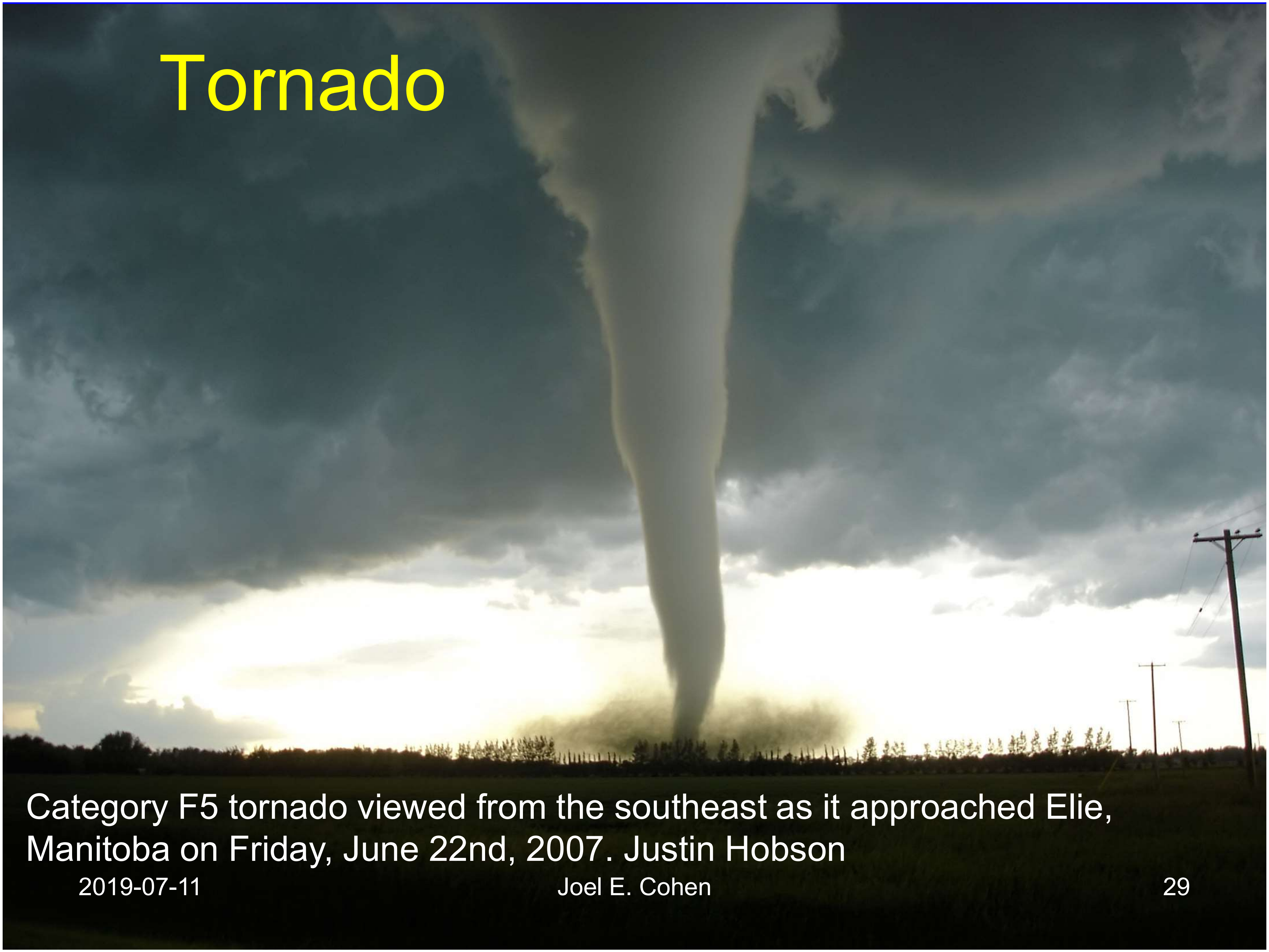
Female β 1960-2009



Male β 1960-2009

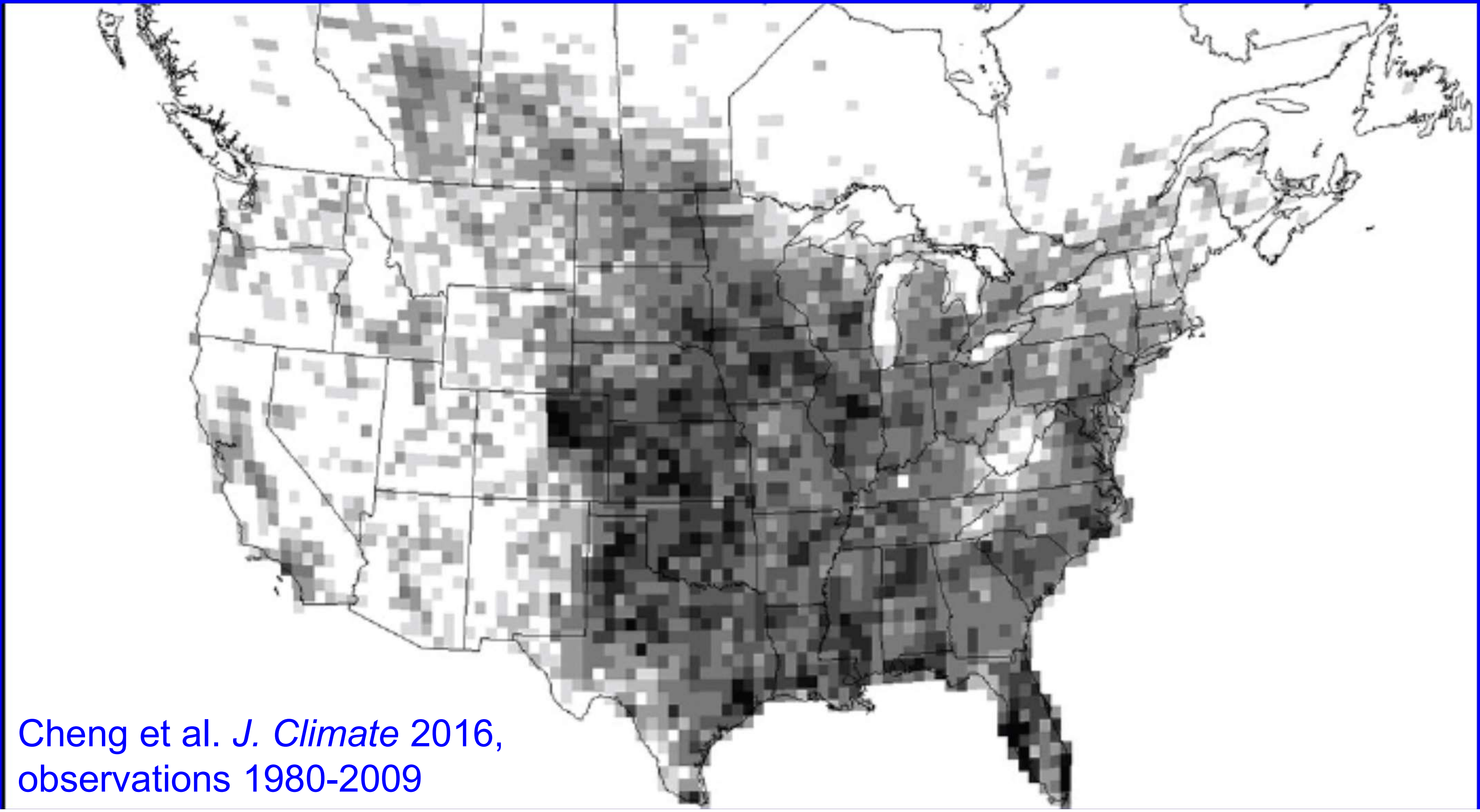


Tornado



Category F5 tornado viewed from the southeast as it approached Elie, Manitoba on Friday, June 22nd, 2007. Justin Hobson

USA has more tornadoes than any other country. (Lloyd's)



Cheng et al. *J. Climate* 2016, observations 1980-2009

Outbreaks (≥6 tornadoes) cause most damage.

Outbreak is defined as ≥6 tornadoes starting ≤6 hours apart.

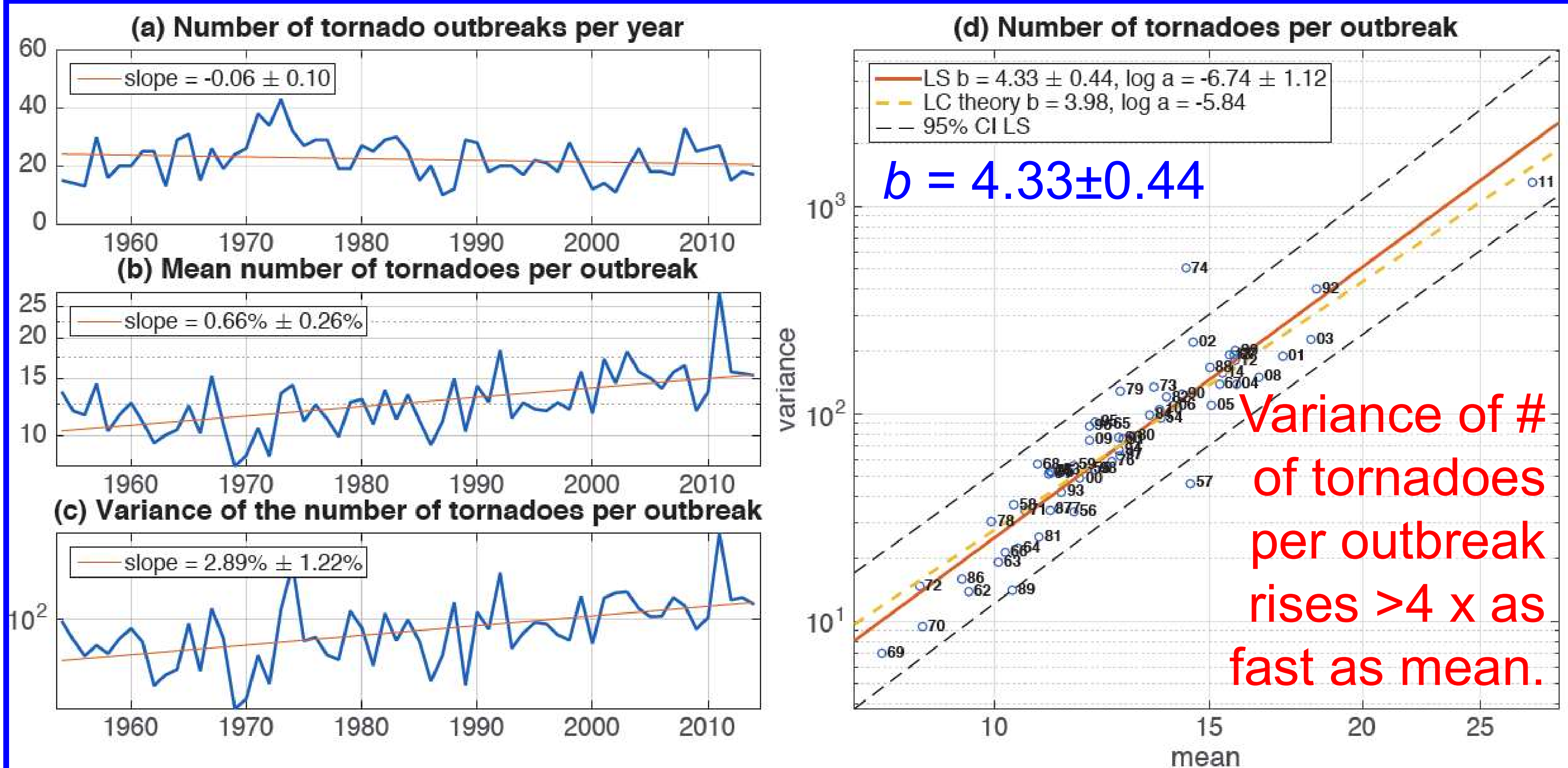
1972–2010: 79% of tornado fatalities & most economic losses occurred in outbreaks.

No trends in numbers of reliably reported tornadoes or outbreaks in last half century.

Mean & variance of number of tornadoes per outbreak, & insured losses, increased significantly in last half century.

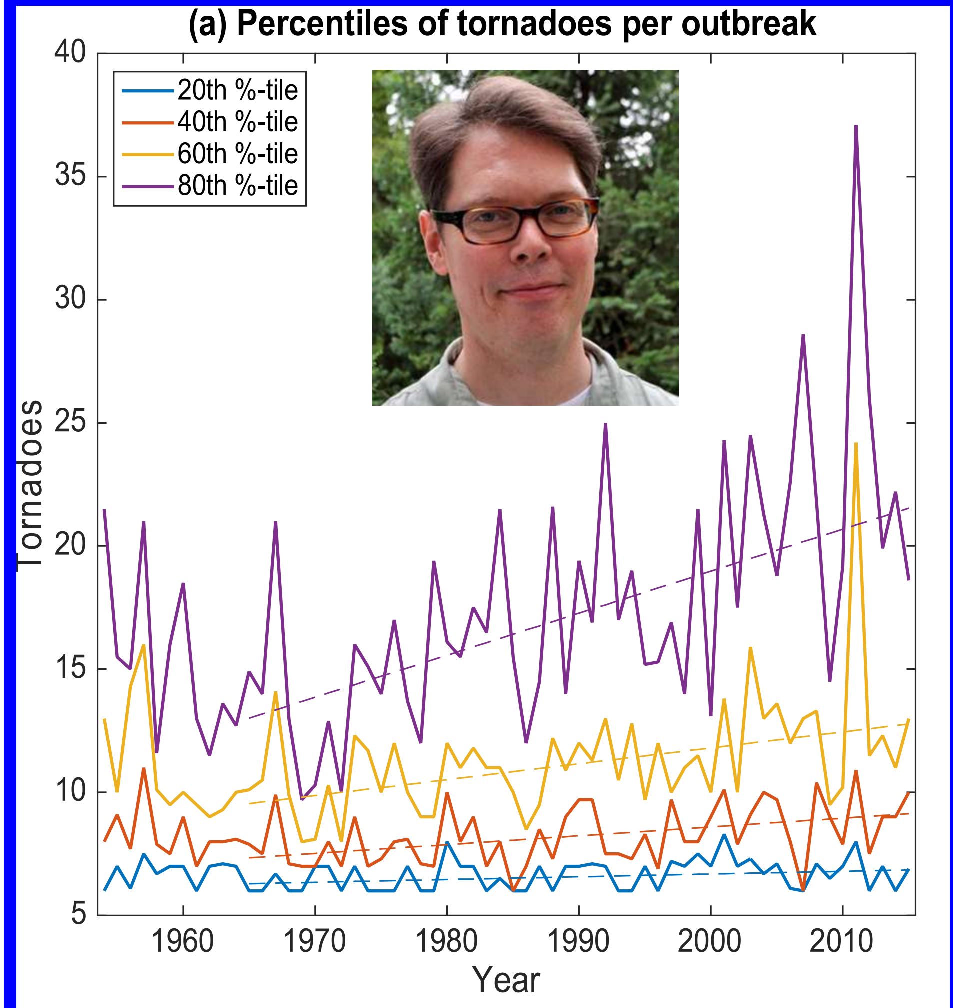
F1+ tornadoes per outbreak in USA: variance~(mean)$^{4.3}$

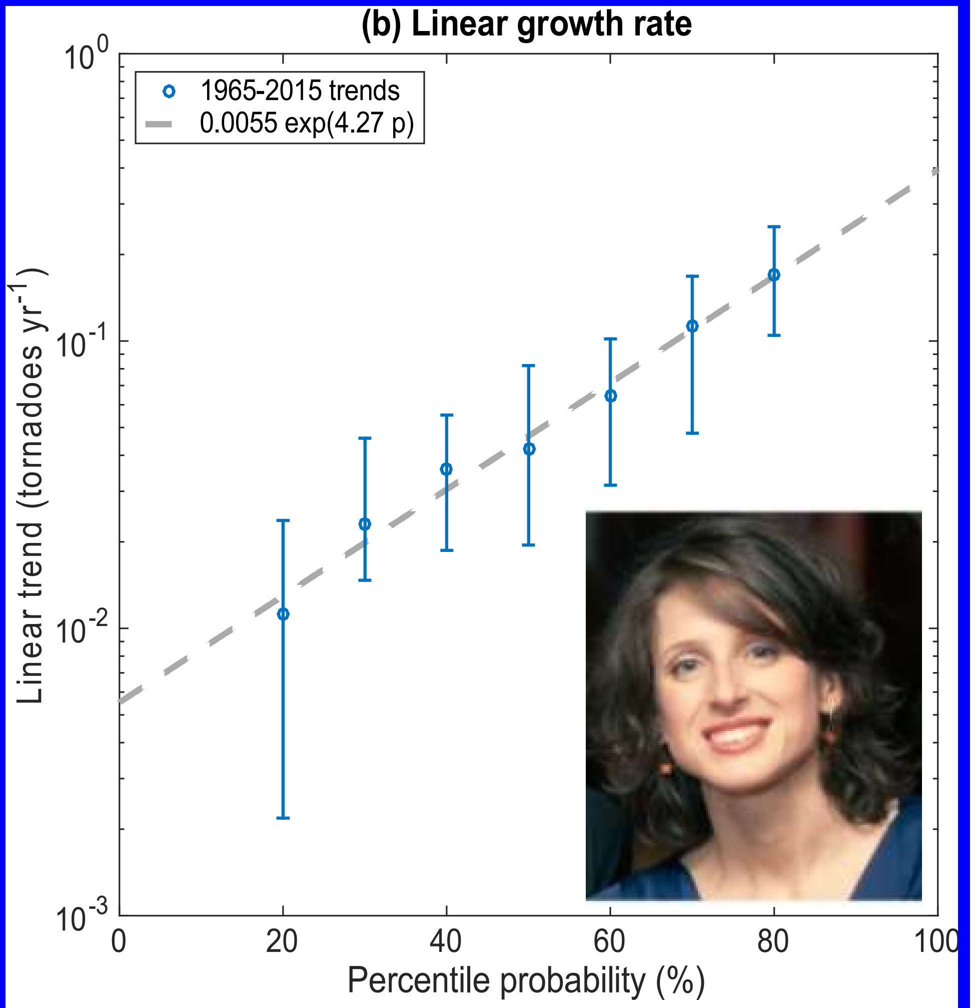
Tippett & Cohen, *Nature Communications* 2016



Higher percentiles increased faster.

"quantile regression"





Tippett, Lepore, Cohen *Science* 2016

Extreme outbreaks (12+ tornadoes) increased extremely.

- "Once in 5 years" extreme outbreak increased from 40 tornadoes in 1965 to 80 tornadoes in 2015.
- "Once in 25 years" extreme outbreak more than doubled from 1965 to 2015.

Tippett, Lepore, Cohen *Science* 2016

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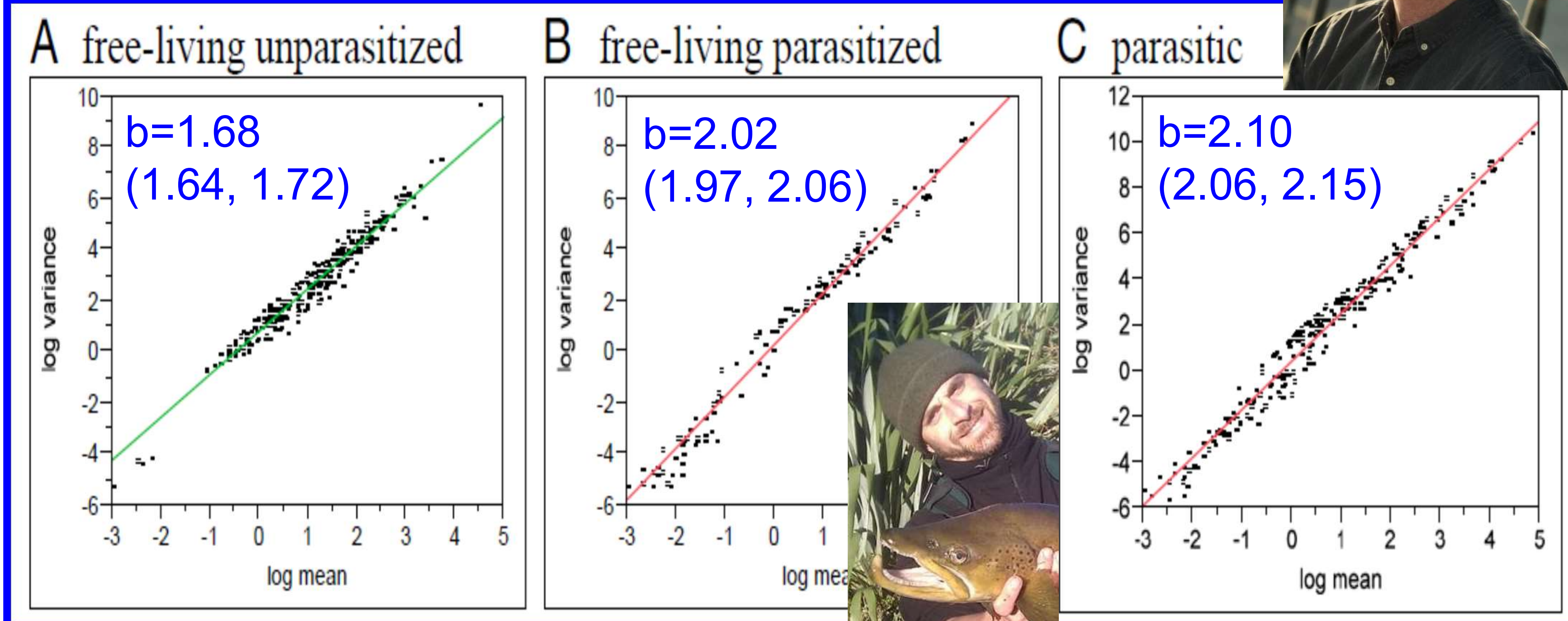
Metazoan population density (individuals m^{-2}) obey spatial TL. Parameters differ by life style.

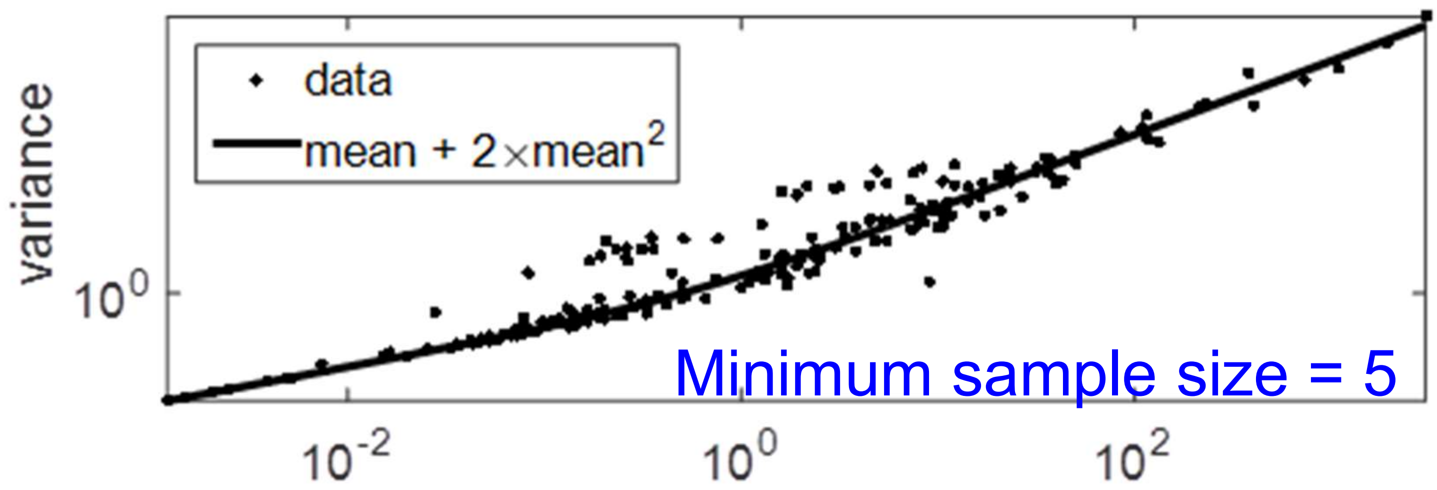
Lagrue, Poulin, Cohen PNAS 2015

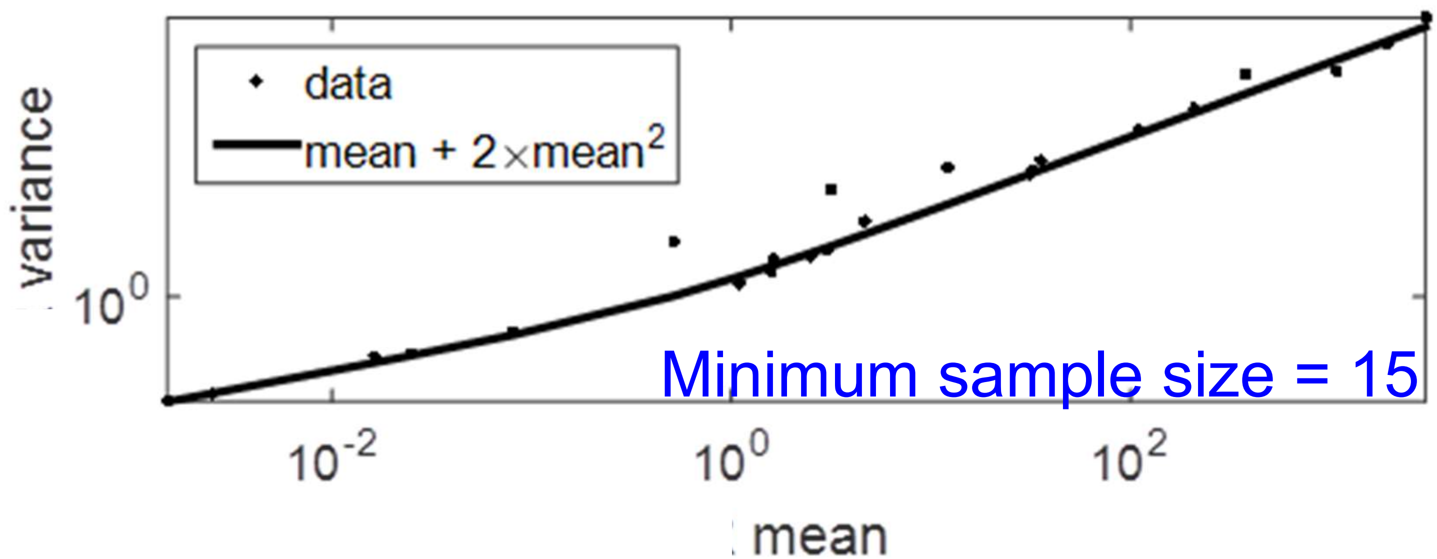
First demonstration that the parameters of TL depend on lifestyle within a given metazoan community



Parasites/host follow negative binomial: variance=mean+2(mean^2).

Cohen, Lagrue, Poulin, *PNAS* 2016





Negative binomial

k "index of aggregation"

var = mean + k*mean^2

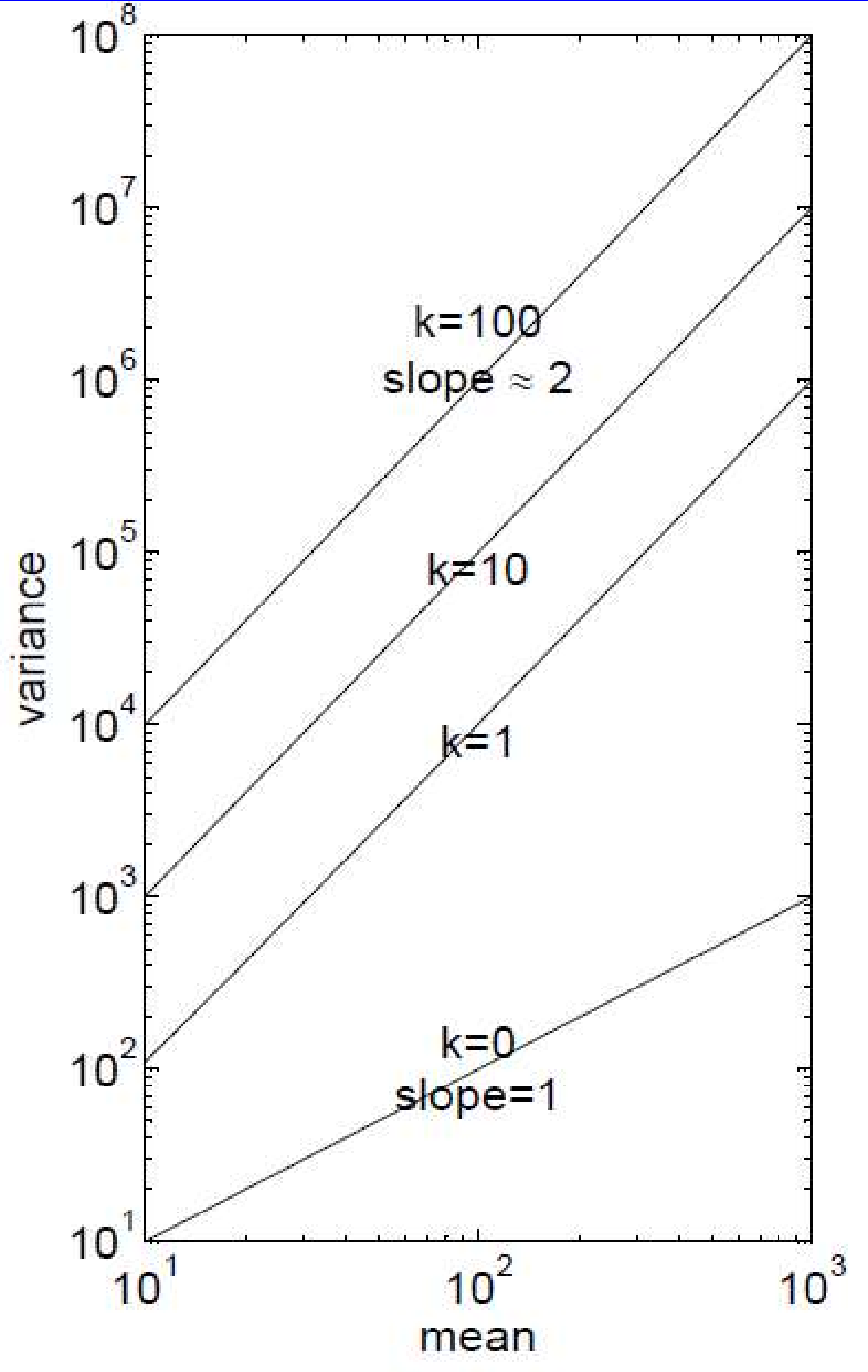
$0 \leq k \leq \infty$

$\Rightarrow$

variance approximates

a*mean^b, $1 \leq b \leq 2$

Bartlett 1947, Hayman & Lowe 1961, Taylor 1961



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5. Conclusions

Theories

Random sampling in blocks from skewed distributions: a "null" model

Cohen & Xu *PNAS* 2015: $b \approx skewness/CV$.

Parametric families of distributions

Exponential change of mean & variance

Nonnegative random variables with infinite mean

Brown, Cohen, de la Peña, *J. Appl. Prob.* 2017

Brown, Cohen & colleagues, *in preparation*

Primes and twin primes

Cohen, *The American Statistician* 2016

Random sampling of skewed distributions

100x100 matrices of iid values from a fixed distribution:

A Poisson

B negative binomial

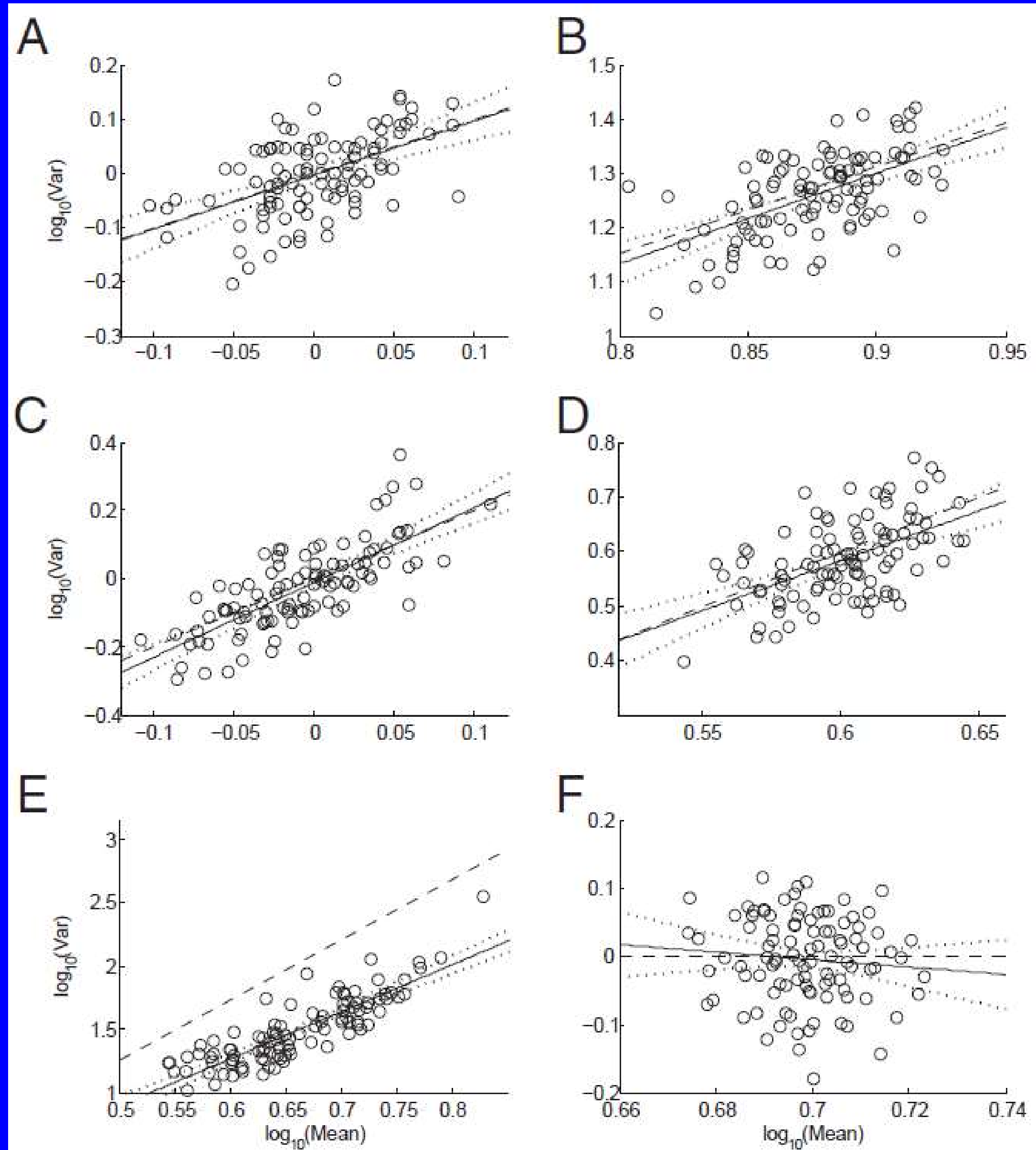
C exponential

D gamma

E lognormal

F normal translated

Cohen & Xu *PNAS* 2015



2019-07-11

A "null model" of Taylor's law: random sampling in blocks

Cohen & Xu *PNAS* 2015

Consider $N > 2$ "blocks" (samples) of iid observations (random samples) of r.v. $X \geq 0$ with four finite moments.

Assume block j has $n_j > 3$ observations, $j = 1, \ldots, N$.

Total number of observations is $n_1 + \cdots + n_N$.

Assume X has 4 moments.

Suppose X has finite mean $E(X) = M > 0$,
finite variance $var(X) = V > 0$,
finite 3d & 4th central moments
$E([X - M]^h) = \mu_h$, $h = 3, 4$,
coefficient of variation $CV = V^{1/2}/M$,
skewness $\gamma_1 = \mu_3/V^{3/2}$,
kurtosis $\kappa = \mu_4/V^2$.

Least-squares regression of log sample mean & sample variance

Let x_{ij} denote observation i in block j, $i = 1, \ldots, n_j$.

Assume all observations x_{ij} are iid,

& n_j is large enough that $m_j > 0, v_j > 0$,

& $\hat{b}$, $\widehat{\log(a)}$ denote the least-squares estimators, respectively, of b, $\log(a)$ in log-log linear TL,

$\log(v_j) = \log(a) + b \times \log(m_j),\ \ j = 1, \ldots, N.$

Then, for large N, large n_j,

Slope *b* of TL ∝ skewness

Skewness is $\gamma_1 = \mu_3/V^{3/2}$,

kurtosis is $\kappa = \mu_4/V^2$.

$$\hat{b} \to_P \frac{cov(m_j, v_j)}{MV} \Big/ \frac{var(m_j)}{M^2} = \mu_3 M/V^2 = \gamma_1/CV\,.$$

$$(N-2)s^2(\hat{b}) \to_P M^2(\mu_4 V - V^3 - \mu_3^2)/V^4$$

$$= (\kappa - 1 - \gamma_1^2)/(CV)^2\,.$$

$s^2(\hat{b}) \geq 0$ implies $\kappa - 1 - \gamma_1^2 \geq 0$.

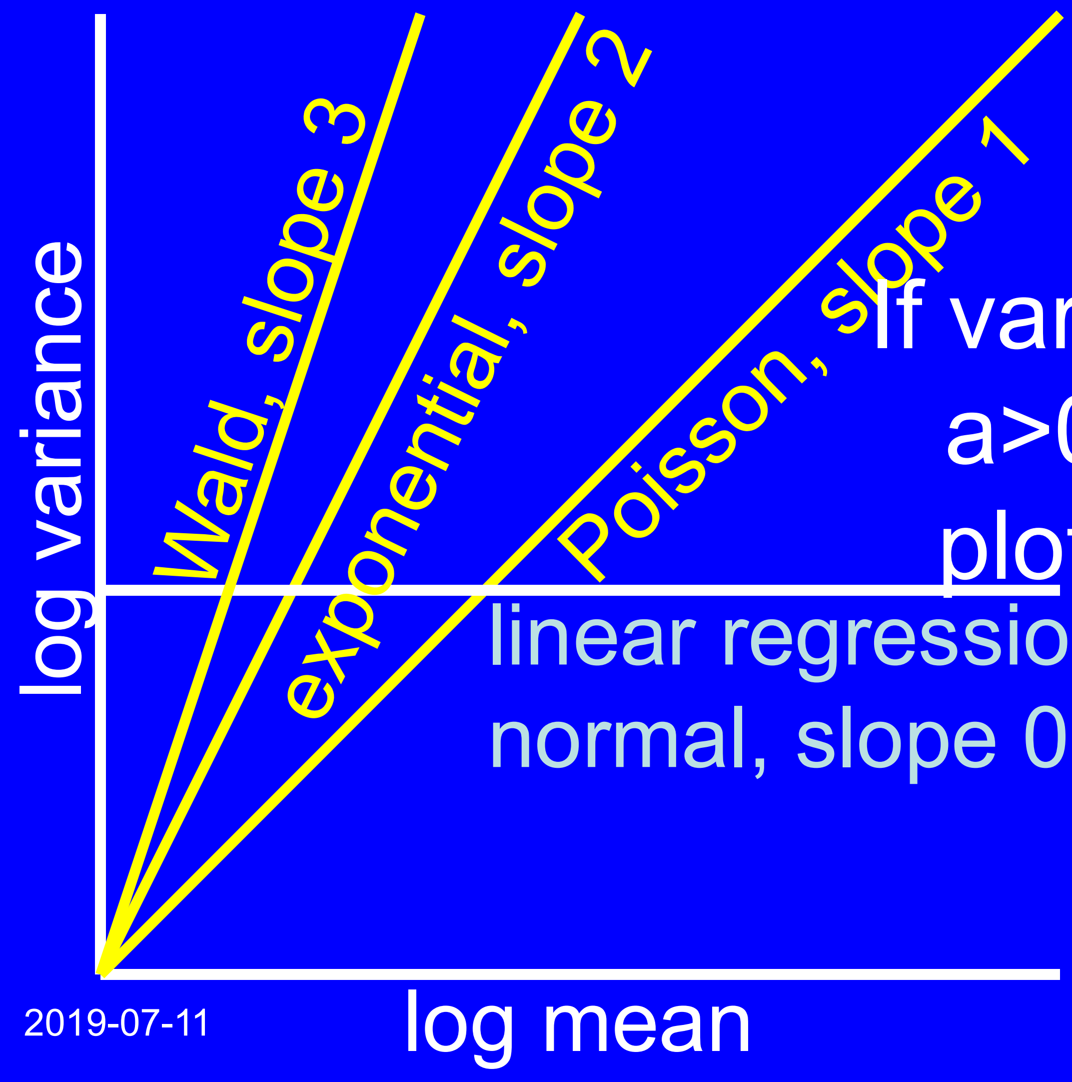
Rohatgi & Székely, *Stat. & Probab. Letters* 1989

$\widehat{\log(a)} \to_P \log V - \frac{\gamma_1}{CV} \cdot \log M$.

Test the null model: are blocks iid?

- Test for homogeneity (equality) of variances
  - Bartlett's test if data are normal Shapiro-Wilk test
  - Levene's test if data are not normal
  - Brown & Forsythe 1974 use median or trimmed mean in addition to mean
- Test for homogeneity (equality) of means (IF homogeneity of variances is NOT rejected)
  - 1-way analysis of variance
  - Kruskal-Wallis nonparametric test

Taylor's law holds for some classical probability distributions.



If variance=a(mean)^b, a>0, then on log-log plot, TL has slope b, intercept log(a). Often $1 \le b \le 2$.

Exponential mean & variance

If mean $m(t)$ & variance $v(t)$ are exponential functions of parameter *t*,
$m(t) = \alpha e^{\beta t}, v(t) = \gamma e^{\delta t}, \alpha > 0, \beta \neq 0, \gamma > 0,$

then $v(t) = am(t)^b$ with $b = \delta/\beta$.

If mean & variance are *asymptotically* exponential functions of time *t,* & mean is invertible, then variance is *asymptotically* a power-law function of mean.

Generally, if $m(t) = f(t) > 0, v(t) = a \cdot f(t)^b$, then $v(t) = a \cdot [m(t)]^b$.

Exponential parameterizations of mean & variance

Branching processes Cohen *Theoret. Pop. Biol.* 2014

Birth-and-death processes Cohen *T.P.B.* 2014

Multiplicative random walk (iid factors)
Cohen, Xu, Schuster, *Proc. R. Soc. B* 2013

Markovian multiplicative process
Cohen, *Theoret. Pop. Biol.* 2014

Deterministic exponential clones
Cohen, *Theoret. Pop. Biol.* 2013

What happens to sample TL when mean is infinite?


Mark Brown


Victor de la Peña

Stable laws with support [0,∞)

For $c > 0, 0 < \alpha < 1$, define $X =_d F(c, \alpha)$ iff $\Pr\{0 \le X < \infty\} = 1$ &

for any $s \ge 0$, Laplace tr. $Ee^{-sX} = e^{-(cs)^{\alpha}}$.

X is a stable law with "tail index" α.

X has infinite mean.

All stable laws with support [0,∞) have this form. Any nonnegative r.v. Y in the domain of attraction of X obeys $\Pr\{Y > x\} = x^{-\alpha}L(x)$, where $L(x)$ is a function slowly varying at ∞.

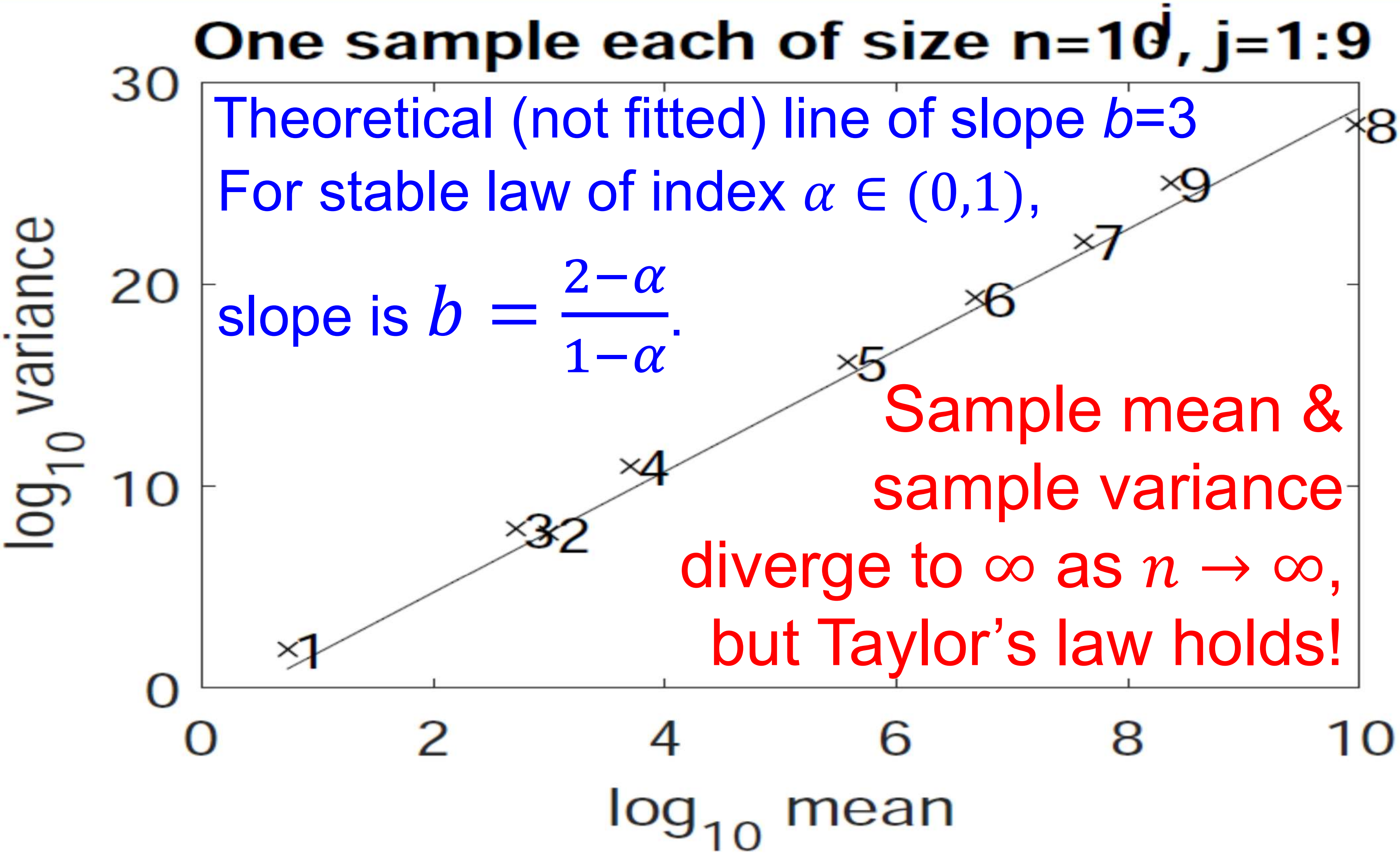
Easy case: stable law with $\alpha = \frac{1}{2}$

$F(c, \frac{1}{2}) =_d c/(2Z^2)$ where $Z =_d \mathcal{N}(0,1)$.

$F(c, \frac{1}{2})$ is the distribution of the first passage time to $\sqrt{\frac{c}{2}}$ in standard Brownian motion.

$F(c, \alpha), 0 < \alpha \neq 1/2 < 1$, has no familiar, tractable expression.

Stable law with $\alpha = 1/2$ obeys TL with increasing sample sizes.



Asymptotic TL for stable laws (1)

W.o.l.o.g., set $c = 1$.

Let $X_1, \ldots, X_n =_d F(1, \alpha)$ iid, $n > 1$.

Define $b := \frac{2-\alpha}{1-\alpha}$, $W_n(b) := \frac{s_n^2}{m_n^b}$. Then

1. $\lim_{n\to\infty} EW_n(b) = \lim_{n\to\infty} E\left(\frac{s_n^2}{m_n^b}\right) = 1 - \alpha$;

(a new form of TL with non-degenerate limit distribution!)

2. $Var\big(W_n(b)\big) = (1-\alpha)^2\left(1 + \frac{2\alpha}{n-1}\right) \to$
$(1-\alpha)^2$ as $n \to \infty$;

3. $\sup_n \left\{E\big(W_n(b)\big)^k\right\} < \infty$ for all $k \geq 1$.

Asymptotic TL for stable laws (2)

4. $W_n(b) := \frac{s_n^2}{m_n^b} \to_d W(b)$ &

$$\forall k \geq 1, \lim_{n\to\infty} E\big(W_n(b)\big)^k = E\big(W(b)\big)^k < \infty;$$

5. $E\left[(\log W(b))^2\right] < \infty;$

6. $\log s_n^2 = b \log m_n + E \log W(b) + e$, where

$$Ee = 0, Var(e) = Var(\log W(b)) < \infty;$$

7. $\frac{\log s_n^2}{\log m_n} \to_p b := \frac{2-\alpha}{1-\alpha}.$

A wider family of heavy-tailed distributions

If, as $x \to \infty$, $\frac{\Pr\{X>x\}}{[L(x)x^{-\alpha}]} \to 1$ &

the slowly varying function $L(x) \to \frac{1}{\Gamma(1-\alpha)}$,

then $W_n(b) := \frac{s_n^2}{m_n^b} \to_d W(b)$, $b := \frac{2-\alpha}{1-\alpha}$.

This limiting r.v. $W(b)$ is the same $W(b)$ as when X is stable with index α.

The set of r.v.s that satisfy assumptions above are a family of heavy-tailed distributions, parameterized by $L(x)$, that obeys TL.

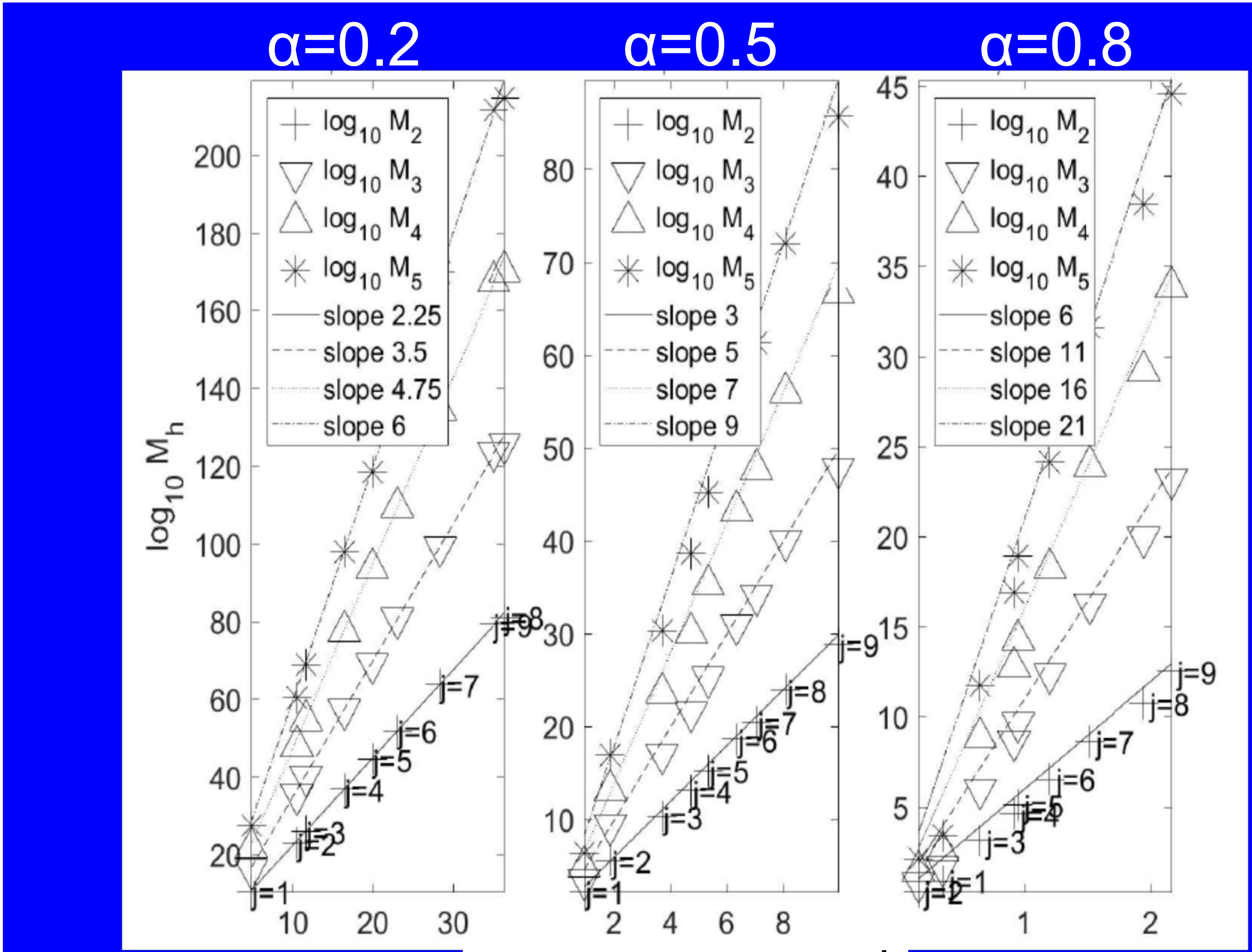
Nonnegative stable laws in progress

Sample higher moments

Sample higher central moments

Upper & lower sample semimoments,
upper & lower sample semivariances

Mark Brown, Cohen, colleagues


α=0.2
α=0.5
α=0.8
log10 Mh
log10 M2
log10 M3
log10 M4
log10 M5
slope 2.25
slope 3.5
slope 4.75
slope 6
slope 3
slope 5
slope 7
slope 9
slope 6
slope 11
slope 16
slope 21
j=1
j=2
j=3
j=4
j=5
j=6
j=7
j=8
j=9

$\log m_n,\, n = 10^j$

Primes

Primes not exceeding x = 10 are
$p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7$.

The number $\pi(x)$ of primes not exceeding x is $\pi(x) = 4$.

The mean of those 4 primes is

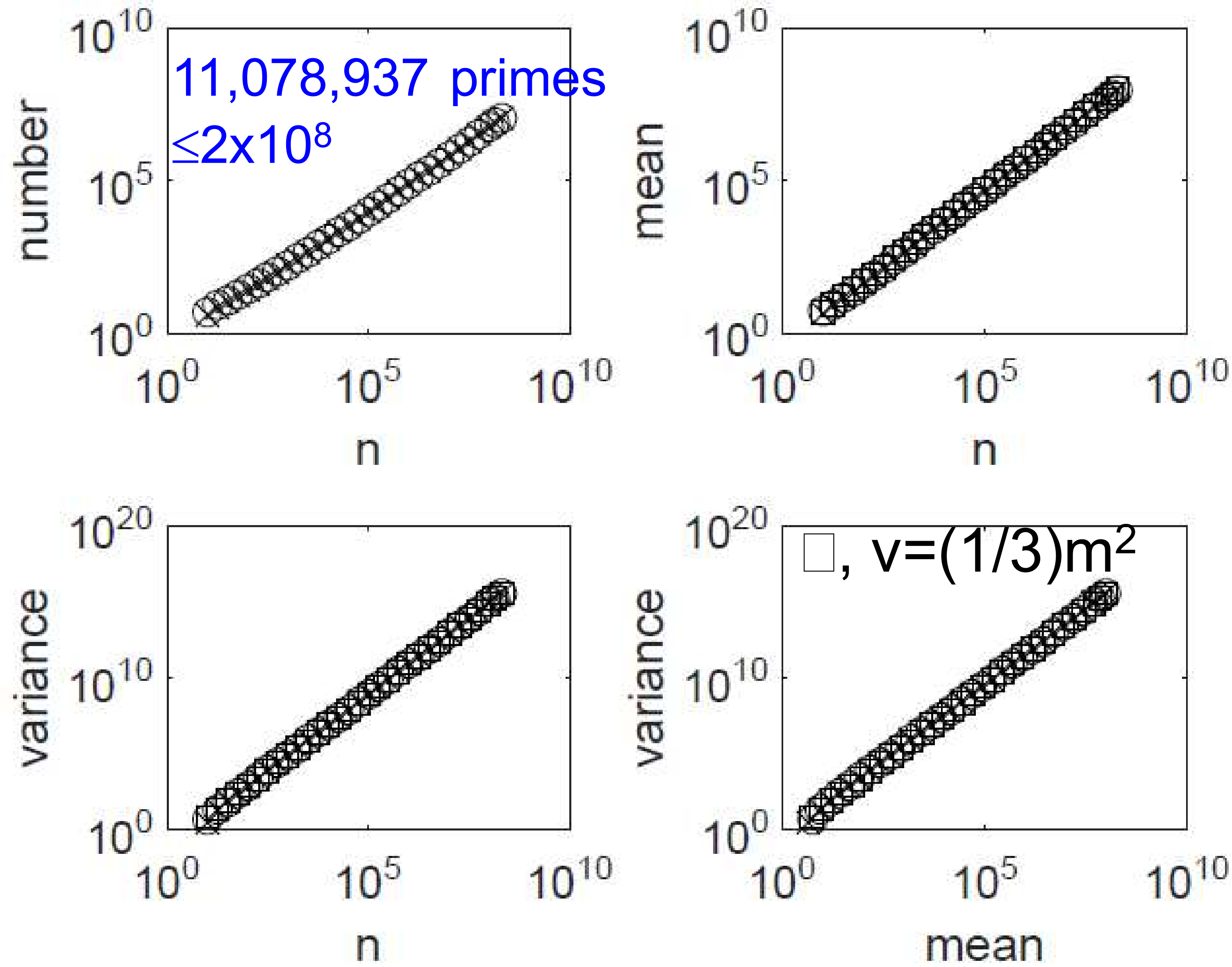
$m(x) = 4.25 = (2 + 3 + 5 + 7)/4$.

The variance of those 4 primes is $v(x) = 3.6875 = (2^2 + 3^2 + 5^2 + 7^2)/4 - 4.25^2$.

Primes obey $v(x) \sim \frac{1}{3} m(x)^2$ as $x \to \infty$.

x, exact enumeration
o, logarithmic integral
□, asymptotic

Cohen, *The American Statistician* 2016



2019-07-11

Twin primes

A natural number *p* is defined to be a twin prime iff *p* is a prime and *p*+2 is a prime.

3 and 5 are all the twin primes not exceeding 10. 7 is not a twin prime because 7 + 2 = 9 is not a prime.

No one knows whether the number of twin primes is finite or infinite.

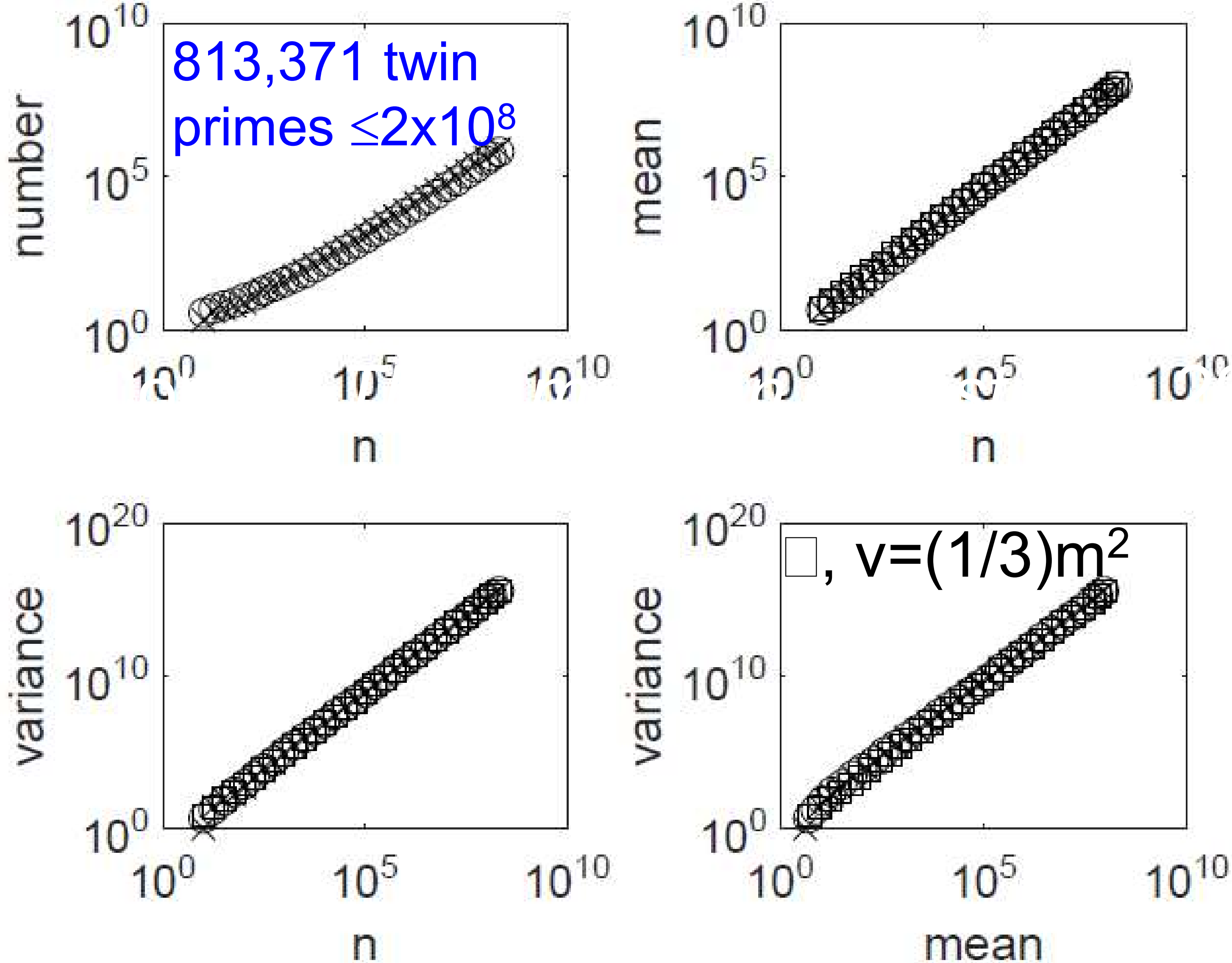
Mean twin prime <10 is $m(10) = (3+5)/2 = 4$.

Variance is $v(10)=(1+1)/2=1$.

Twin primes obey $v(x) \sim \frac{1}{3} m(x)^2$ as $x \to \infty$, given Hardy-Littlewood (1923) twin-prime conjecture.

x, exact enumeration
o, logarithmic integral
□, asymptotic

Cohen, *The American Statistician* 2016



Prime number theorem

If $f(x)$ and $g(x)$ are real-valued functions of a real x, then $f(x) \sim g(x)$ iff, as $x \to \infty$, we have $f(x) \to \infty$, $g(x) \to \infty$ & $f(x)/g(x) \to 1$.

Hadamard & de la Vallée Poussin, 1896:

$$\pi(x) \sim \mathrm{li}(x) \equiv \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}.$$

Primes are asymptotically uniform on [2, *x*]. Cohen, *TAS* 2016

Let $X(x)$ be r.v., $X(x) = p \in \mathbb{P}(x)=\{\text{primes} \leq x\}$ with probability $1/\pi(x)$, where $\pi(x)=\#\mathbb{P}(x)$.

For $r \in [0,1]$, the cdf of $X(x)$ is defined as $F_{X(x)}(rx) = \Pr\{X(x) \leq rx\}$.

Theorem:

For any $r \in [0,1]$, as $x \to \infty$, $F_{X(x)}(rx) \sim r$.

Proof. $F_{X(x)}(rx) = \frac{\#\mathbb{P}(rx)}{\#\mathbb{P}(x)} = \frac{\pi(rx)}{\pi(x)} \sim \frac{(rx)/\log(rx)}{x/\log x}$

$= \frac{r \log x}{\log x + \log r} \to r$ as $x \to \infty$. QED

Hardy-Littlewood Twin Prime Conjecture

HLTPC: for some constant $C_2 > 0$, for all $x \in [2, \infty)$, $\#\mathbb{P}_2(x)$ = #{3, 5, 11, 17, … $\leq x$} = $\pi_2(x) \sim 2C_2 \mathrm{li}_2(x) \equiv 2C_2 \int_2^x \frac{dt}{(\log t)^2} \sim 2C_2 \frac{x}{(\log x)^2}$.

$C_2 \approx 0.6601618158 \cdots$ Sebah & Gourdon 2002

Theorem Cohen, TAS 2016. IF $\pi_2(x) \sim 2C_2 \frac{x}{(\log x)^2}$,

then twin primes are asymptotically uniform on [2,x].

Primes satisfy asymptotic TL.

Corollary. Variance $v(x)$ & mean $m(x)$ of primes $\leq x$ obey TL asymptotically as $x \to \infty$:

$$v(x) \sim (1/3)\big(m(x)\big)^2.$$

IF HLTPC holds,

variance & mean of twin primes $\leq x$ obey same TL asymptotically as $x \to \infty$.

Cohen, *TAS*, 2016

Hardy-Littlewood, Bunyakovsky, Schinzel's H, Bateman-Horn (1962) conjectures on admissible prime configurations

(5,7,11), (11,13,17), …, (*p*, *p*+2, *p*+6): *n*=3

Among positive integers ≤*x*, the number of prime-generating integers for *n* polynomials under reasonable conditions is

$$P(x) \sim \frac{C}{D} \int_2^x \frac{dt}{(\log t)^n}.$$

The mean *m*(*x*) & variance *v*(*x*) of these integers obey TL: $v(x) \sim \frac{1}{3} m(x)^2$ as $x \to \infty$.

Outline

1. What is Taylor's law (TL)?
2. Empirical examples from my work
3. TL does not always hold
4. Theories of TL
5. ➔ Conclusions

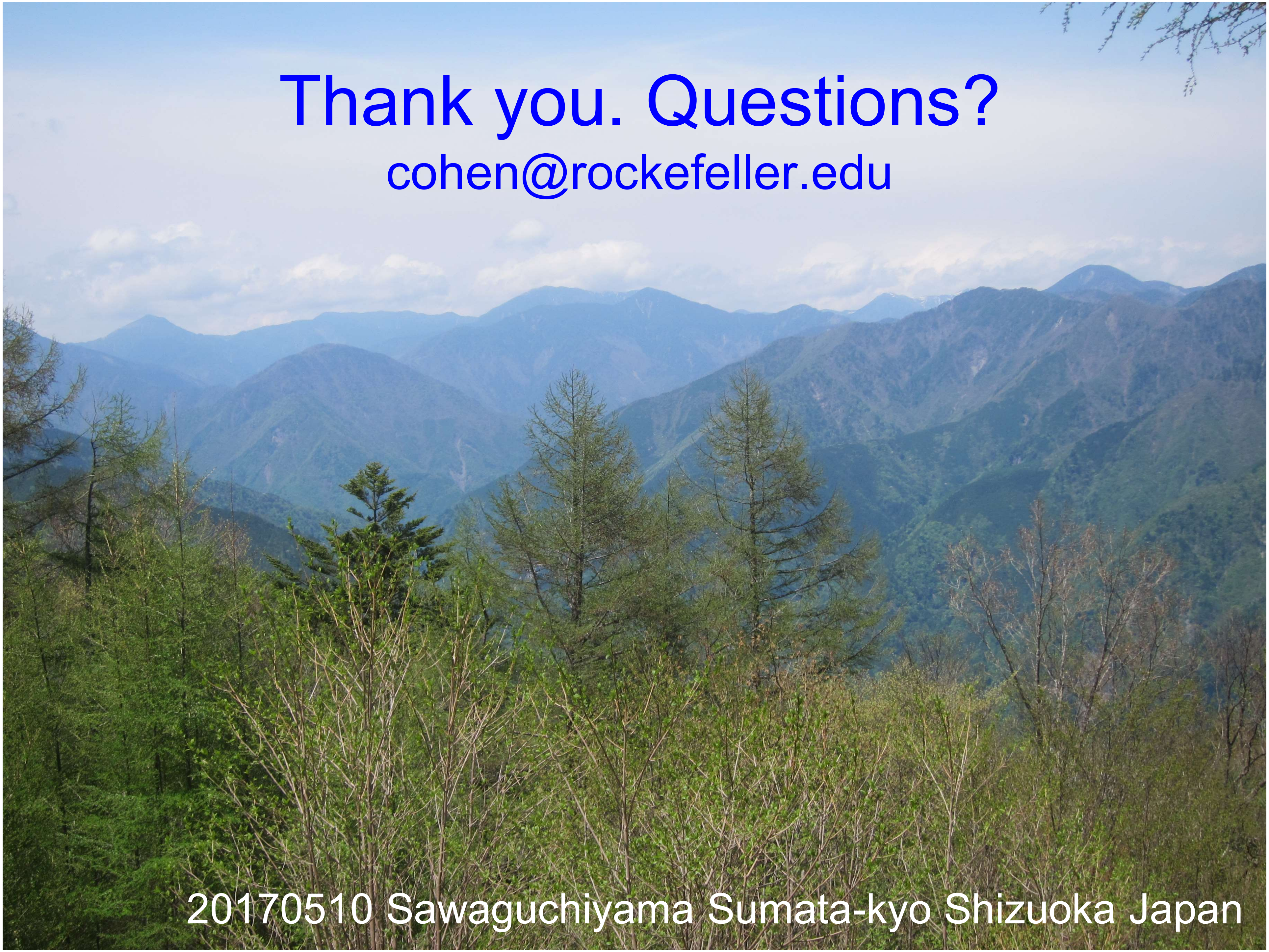
Conclusions: many roads lead to TL.

Many models yield TL exactly or asymptotically.

Power-law form & parameter values of TL do not determine underlying mechanisms.

If the underlying mechanism is known, parameter values of TL can discriminate modes of operation of the mechanism.

Interpreting the parameters of TL in terms of a specific mechanism requires testing the assumptions against detailed data.


Thank you. Questions?
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