Stochastic Processes and their Applications 41 (1992) 261-271
North-Holland

Random arithmetic-geometric means and random pi: Observations and conjectures

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Received 24 September 1990

Two random versions of the arithmetic-geometric mean of Gauss, Lagrange and Legendre are defined. Almost sure convergence and nondegeneracy are proved. These random arithmetic-geometric means in turn define two random versions of π. Based on numerical simulations, inequalities and equalities are conjectured. A special case is proved. Further proofs are invited.

AMS Subject Classifications (1985 revision): Primary 60J05; Secondary 65D20.

nonlinear iteration * elliptic integrals * pi * Markov processes with discrete parameter

1. Introduction

Let $a = a(0)$ and $b = b(0)$ be two distinct positive real numbers, and suppose $a(0) > b(0)$. Let n run through the nonnegative integers. Define

$$a(n+1) = \tfrac{1}{2}[a(n) + b(n)], \qquad b(n+1) = [a(n)b(n)]^{1/2}. \tag{1.1}$$

It was known in the eighteenth century that $\lim_{n\uparrow\infty} a(n)$ and $\lim_{n\uparrow\infty} b(n)$ both exist and that the limits are equal. This common limit $M(a(0), b(0))$ is called the arithmetic-geometric mean (AGM) of $a(0)$ and $b(0)$.

The AGM is useful in computing elliptic integrals, π, and many other mathematical quantities. See Cox (1984, 1985), Arazy et al. (1985), and Borwein and Borwein (1987) for recent reviews. Specifically, the AGM makes it possible to compute elliptic

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* Supported in part by U.S. National Science Foundation grants BSR 84-07461 and BSR 87-05047. This author did part of the work at the Department of Mathematical Sciences, T.J. Watson Research Center, IBM, Yorktown Heights, NY 10598 (1986), at the Institute for Advanced Study, Olden Lane, Princeton, NJ 08540 (1989), and at the Department of Biophysics, Kyoto University, Japan (1990), and acknowledges with thanks the hospitality of Mr. and Mrs. William T. Golden.

** Supported in part by U.S. National Science Foundation grant DMS 86-01800.

integrals rapidly (e.g., Cox 1984, 1985; Borwein and Borwein 1987) via the classic formula, for $0<a,\ b<\infty$,

$$M(a, b)=\frac{\pi}{2}\left[\int_0^{\pi/2}[a^2\cos^2 t+b^2\sin^2 t]^{-1/2}\,\mathrm{d}t\right]^{-1}. \tag{1.2}$$

The AGM can also be used to compute π to very high accuracy, using the formula (Salamin 1976, p. 567; Cox 1985, p. 148)

$$\pi=2\frac{[M(2^{1/2}, 1)]^2}{1-\sum_{n=1}^{\infty}2^n f(n)}, \tag{1.3}$$

where $f(n)=[a(n)]^2-[b(n)]^2$. (Cox (personal communication, 9 January 1987) pointed out that he (1985) mistakenly gave the exponent of 2 in the summation as $n+1$, but it should be n.)

We consider two random versions of the iteration (1.1). Let U be a random variable uniformly distributed on the interval $[0, 1]$. Let $U(n)$, $n=0, 1, \ldots$, be an infinite sequence of independently and identically distributed (i.i.d.) copies of U. Define the sequences $\{A(n)\}$ and $\{B(n)\}$ of random variables by $A(0)=a(0)$, $B(0)=b(0)$ almost surely (a.s.) and

$$\begin{aligned} A(n+1)&=U(n)A(n)+[1-U(n)]B(n),\\ B(n+1)&=A(n)^{U(n)}B(n)^{1-U(n)}. \end{aligned} \tag{1.4}$$

In (1.4), an independent random value $U(n)$ is chosen at each step. The vector $(A(n), B(n))$ is a bivariate Markov process with discrete parameter. Define the sequences $\{C(n)\}$ and $\{D(n)\}$ of random variables by $C(0)=a(0)$, $D(0)=b(0)$ a.s. and

$$C(n+1)=UC(n)+[1-U]D(n), \qquad D(n+1)=C(n)^U D(n)^{1-U}. \tag{1.5}$$

In (1.5), a single value of U is used for every step of the iteration. From a knowledge of $(C(0), D(0))$ and $(C(1), D(1))$, U can be determined exactly and hence the future of (1.5) can be predicted, though this is not true from a knowledge of only $(C(n), D(n))$ for any single value of n.

By analogy with similar models in statistical mechanics, (1.4) may be called the annealed AGM process and (1.5) may be called the quenched AGM process. Since both processes reduce to (1.1) when all the random parameters $U, U(0), U(1), \ldots$ are replaced by their common expectations $\frac{1}{2}$, (1.1) may be thought of as a mean field theory for both the quenched and the annealed AGM processes.

Let $X(a, b)$ denote the common limit of $A(n)$ and $B(n)$ in (1.4), and $Y(a, b)$ likewise for (1.5). (See Proposition 2.1 below.) It is natural to define annealed and quenched random variables analogous to π by

$$\pi_{\mathrm{A}}=2\frac{[X(2^{1/2}, 1)]^2}{1-\sum_{n=1}^{\infty}2^n F(n)}, \tag{1.6}$$

where $F(n)=[A(n)]^2-[B(n)]^2$, and

$$\pi_Q = 2\frac{[Y(2^{1/2},1)]^2}{1-\sum_{n=1}^{\infty}2^n G(n)}, \tag{1.7}$$

where $G(n)=[C(n)]^2-[D(n)]^2$.

We will find numerically, and in some cases prove mathematically, that there are inequalities between the mean annealed or quenched AGM and the original AGM, as well as inequalities in the opposite direction between the mean π_A or mean π_Q and π. If methods could be developed to prove such particular inequalities, they might be a step towards dealing with the large number of other functionals of the AGM that have been studied in the classical deterministic situation. More generally, nonlinear iterations with random parameters, such as (1.4) and (1.5), seem likely to play an important role in an eventual marriage of nonlinear dynamical systems and stochastic processes. It seems worthwhile to consider some examples that move toward such a marriage.

2. Preliminary results

Proposition 2.1. *For every sample path,* $\lim_{n\uparrow\infty} A(n)$, $\lim_{n\uparrow\infty} B(n)$, $\lim_{n\uparrow\infty} C(n)$, *and* $\lim_{n\uparrow\infty} D(n)$ *exist. Moreover,*

$$\lim_{n\uparrow\infty} A(n) = \lim_{n\uparrow\infty} B(n) \quad \textit{a.s.}, \tag{2.1}$$

$$\lim_{n\uparrow\infty} C(n) = \lim_{n\uparrow\infty} D(n) \quad \textit{a.s.} \tag{2.2}$$

Proof. For any $u\in[0,1]$, $v=1-u$, $a>0$, and $b>0$, $\log(ua+vb)\geqslant u\log a+v\log b$, with strict inequality unless $a=b$ or $u=0$ or $u=1$, because log is strictly concave. Taking exp of both sides of this inequality gives $ua+vb\geqslant a^u b^v$, so $\max(a,b)\geqslant ua+vb\geqslant a^u b^v\geqslant\min(a,b)$. Thus $A(n)$ and $C(n)$ are monotone nonincreasing sequences bounded below by $b(0)$, so their limits exist. $B(n)$ and $D(n)$ are monotone nondecreasing sequences bounded above by $a(0)$, so their limits exist. Now if $a_1=ua+vb$ and $b_1=a^u b^v$, then $(a_1-b_1)/(a-b)\leqslant u$. Since $U(n)<\frac{9}{10}$, say, infinitely often in almost every (a.e.) sample path of the annealed process, (2.1) holds a.s. Since $U<1$ a.s. in the quenched process, (2.2) holds a.s. $\square$

A slight extension of this proposition is proved by Nussbaum (1990, his Corollary 2.2, p. 451).

Let the random variable $X=X(a(0),b(0))$ be the a.s. common value of the limits $\lim_{n\uparrow\infty} A(n)=\lim_{n\uparrow\infty} B(n)$, and let $Y=Y(a(0),b(0))$ be the a.s. common value of the limits $\lim_{n\uparrow\infty} C(n)=\lim_{n\uparrow\infty} D(n)$. X and Y are, respectively, the annealed and the quenched AGMs of $a(0)$ and $b(0)$.

Proposition 2.2. *The distributions of X and Y are nondegenerate.*

Proof. Since $A(1) \geqslant X \geqslant B(1)$ and $C(1) \geqslant Y \geqslant D(1)$, it suffices to show that $P[B(1) > a(1)] > 0$ and $P[A(1) < a(1)] > 0$ because $P[B(1) > a(1)] = P[C(1) > a(1)]$ and $P[A(1) < a(1)] = P[C(1) < a(1)]$. It is elementary to see that

$$B(1) > a(1) \quad \text{if and only if} \quad U(0) > \log\left[\frac{1}{2}\left(1 + \frac{a(0)}{b(0)}\right)\right] \Big/ \log\frac{a(0)}{b(0)} \tag{2.3}$$

and since $a(0) > b(0)$, the quantity on the right is greater than 0 and less than 1. Hence the events in (2.3) occur with positive probability. Similarly,

$$A(1) < a(1) \quad \text{if and only if} \quad U(0) < \tfrac{1}{2} \tag{2.4}$$

and the probability of the events in (2.4) is $\frac{1}{2}$. □

Proposition 2.3.

$$E[A(n+1) \mid A(n) = a(n), B(n) = b(n)] = a(n+1) \tag{2.5}$$

and

$$E[B(n+1) \mid A(n) = a(n), B(n) = b(n)] = \frac{a(n) - b(n)}{\log a(n) - \log b(n)} > b(n+1). \tag{2.6}$$

Proof. (2.5) is obvious from (1.1) and (1.4). Because the exponential $[a(n)/b(n)]^u$ is strictly convex in u and

$$E[B(n+1) \mid A(n) = a(n), B(n) = b(n)] = b(n) \int_0^1 [a(n)/b(n)]^u \, du \tag{2.7}$$

the inequality in (2.6) follows. □

3. Conjectures and numerical results

By a leap of faith, Proposition 2.3 suggests:

Conjecture 3.1.

$$E[X(a(0), b(0))] > M(a(0), b(0)) \quad \text{and} \quad E[Y(a(0), b(0))] > M(a(0), b(0)).$$

To investigate Conjecture 3.1 numerically, we chose the same case of (1.1) that Gauss studied numerically in 1799 (see e.g. Cox, 1985), namely, $a(0) = 2^{1/2}$ and $b(0) = 1$. We continued the iteration (1.1) until $|a(n) - b(n)| < \delta$, and the iteration in each sample path of (1.4) or (1.5) until $|A(n) - B(n)| < \delta$ or $|C(n) - D(n)| < \delta$, where $\delta = 10^{-12}$. For (1.1), we obtained $M(2^{1/2}, 1)$ shown in Table 1, which agrees, to the number of figures obtained, with the result given by Cox (1985). We computed 10^6 simulations or sample paths of (1.4), cumulating over sample paths the sums

Table 1

The arithmetic-geometric mean and simulations of the annealed arithmetic-geometric mean process and the quenched arithmetic-geometric mean process, for $a(0)=2^{1/2}$ and $b(0)=1$. Because 10^6 independent simulations were performed for each of the latter two processes, the standard deviation of the sample mean is 10^{-3} times the sample standard deviation given below

Iteration	Arithmetic-geometric means		π and its random analogs	
	Sample mean	Sample s.d.	Sample mean	Sample s.d.
Classical (1.1)	1.198 140 235	0	3.141 592 654	0
Annealed (1.4)	1.201 188 679	0.119 505 917	3.096 424 677	0.612 860 519
Quenched (1.5)	1.201 166 777	0.121 533 413	3.097 495 552	0.623 383 374

of X and X^2, and independently 10^6 sample paths of (1.5), cumulating over sample paths the sums of Y and Y^2. From these sums we computed $\bar{X}$ and $\bar{Y}$, the sample means, as the best estimates of $E(X)$ and $E(Y)$, the sample standard deviations s.d.(X) and s.d.(Y), and the standard deviations of the sample means s.d.$(\bar{X})$ and s.d.$(\bar{Y})$. Since the sample size was 10^6, the latter are simply s.d.$(\bar{X})=10^{-3}$ s.d.(X) and s.d.$(\bar{Y})=10^{-3}$ s.d.(Y). A listing of the APL functions used to carry out these and the following computations is available on request.

Table 1 shows that $\bar{X}$ is larger than $M(2^{1/2},1)$ by more than 25 standard deviations of $\bar{X}$, and the same is true for $\bar{Y}$. Relying on the central limit theorem, we conclude that this difference would have happened by chance alone with a probability that is essentially 0 if $E(X)$ or $E(Y)$ were equal to $M(2^{1/2},1)$. We prove in Theorem 4.2 that Conjecture 3.1 is true for a range of values of $(a(0), b(0))$, and in particular that $E[X(2^{1/2},1)]>M(2^{1/2},1)$.

An unexpected finding in Table 1 is that $\bar{X}-\bar{Y}$ is small compared to the estimated standard deviation of this difference, s.d.$(\bar{X}-\bar{Y})=\{[\text{s.d.}(\bar{X})]^2+[\text{s.d.}(\bar{Y})]^2\}^{1/2}$. This suggests:

Conjecture 3.2. $E[X(a(0), b(0))]=E[Y(a(0), b(0))]$.

Unfortunately, this suggestion is false in general, as we shall prove in Theorem 4.3.

Table 1 gives the value of π computed using (1.3), which is accurate to the number of places given, and the sample means $\bar{\pi}_A$ and $\bar{\pi}_Q$ and sample standard deviations s.d.(π_A) and s.d.(π_Q). These quantities were calculated from the same 10^6 simulated sample paths used to evaluate X and Y. The standard deviations of the sample means are 10^{-3} times the sample standard deviations. The results in Table 1 and leaps of faith give:

Conjecture 3.3. $E[\pi_A]<\pi$ and $E[\pi_Q]<\pi$.

Table 1 suggests that $E[\pi_A]=E[\pi_Q]$, but the analogy of Theorem 4.3 makes us doubt the equality. We propose the equality as a question for resolution, not as a firm expectation.

4. An inequality proved

The primary aim of this section is to prove part of Conjecture 3.1, namely, $E[X(a, b)] > M(a, b)$ for $1 < a/b < 2.68$. This range of values for a and b includes the values $a = 2^{1/2}$, $b = 1$, used in the numerical example. We will also show that Conjecture 3.2 is false.

For fixed $a > b > 0$, let $0 \leq u \leq 1$ and define

$$A(u) = ua + (1-u)b, \qquad B(u) = a^u b^{1-u},$$

$$F(u) = M(A(u), B(u)).$$

Lemma 4.1. *For $1 \leq a/b \leq 2.68$, $F(u)$ is convex in u on $[0, 1]$.*

Proof. Let subscripts denote partial derivatives. Then

$$F'(u) = M_1(A(u), B(u))A'(u) + M_2(A(u), B(u))B'(u),$$

$$\begin{aligned} F''(u) = {} & M_1(A(u), B(u))A''(u) + M_2(A(u), B(u))B''(u) \\ & + M_{11}(A(u), B(u))[A'(u)]^2 + M_{22}(A(u), B(u))[B'(u)]^2 \\ & + 2M_{12}(A(u), B(u))A'(u)B'(u). \end{aligned}$$

Using (1.2), the first derivatives of $M(a, b)$ are

$$\begin{aligned} M_1(a, b) = {} & \frac{\pi}{2}\left[\int_0^{\pi/2} [a^2\cos^2 t + b^2 \sin^2 t]^{-1/2}\,\mathrm{d}t\right]^{-2} \\ & \times \int_0^{\pi/2} \frac{a\cos^2 t\,\mathrm{d}t}{[a^2\cos^2 t + b^2\sin^2 t]^{3/2}}, \\ M_2(a, b) = {} & \frac{\pi}{2}\left[\int_0^{\pi/2} [a^2\cos^2 t + b^2 \sin^2 t]^{-1/2}\,\mathrm{d}t\right]^{-2} \\ & \times \int_0^{\pi/2} \frac{b\sin^2 t\,\mathrm{d}t}{[a^2\cos^2 t + b^2\sin^2 t]^{3/2}}. \end{aligned}$$

Note that

$$aM_1(a, b) + bM_2(a, b) = M(a, b).$$

Therefore, differentiating with respect to a and b, respectively, gives:

$$aM_{11}(a, b) + M_1(a, b) + bM_{12}(a, b) = M_1(a, b),$$

$$aM_{12}(a, b) + M_2(a, b) + bM_{22}(a, b) = M_2(a, b).$$

This implies that

$$aM_{11}(a, b) + bM_{12}(a, b) = 0,$$

$$aM_{12}(a, b) + bM_{22}(a, b) = 0.$$

This, together with $A''(u) \equiv 0$ yields

$$F''(u) = M_2(A(u), B(u))B''(u) + M_{12}(A(u), B(u)) \times \left\{ -\frac{[A'(u)]^2 B(u)}{A(u)} + 2A'(u)B'(u) - \frac{[B'(u)]^2 A(u)}{B(u)} \right\}$$

where

$$A'(u) = a - b, \qquad B'(u) = \left(\log \frac{a}{b}\right) B(u), \qquad B''(u) = \left(\log \frac{a}{b}\right)^2 B(u).$$

Therefore

$$F''(u) = B(u) \left\{ \left(\log \frac{a}{b}\right)^2 M_2(A(u), B(u)) - \frac{M_{12}(A(u), B(u))}{A(u)} \left[(a-b) - A(u) \log \frac{a}{b} \right]^2 \right\}.$$

To simplify notation, write for fixed u:

$$f(t) = \frac{\sin^2 t}{A^2(u) \cos^2 t + B^2(u) \sin^2 t}, \qquad g(t) = \frac{\cos^2 t}{A^2(u) \cos^2 t + B^2(u) \sin^2 t}.$$

Also, let

$$\mu(\mathrm{d}t) = \frac{2\,\mathrm{d}t}{\pi[A^2(u) \cos^2 t + B^2(u) \sin^2 t]^{1/2}}.$$

Then

$$M_2(A(u), B(u)) = [M(A(u), B(u))]^2 B(u) \int_0^{\pi/2} f\,\mathrm{d}\mu$$

and

$$M_{12}(A(u), B(u)) = 2[M(A(u), B(u))]^3 A(u)B(u) \int_0^{\pi/2} f\,\mathrm{d}\mu \int_0^{\pi/2} g\,\mathrm{d}\mu - 3[M(A(u), B(u))]^2 A(u)B(u) \int_0^{\pi/2} fg\,\mathrm{d}\mu.$$

Therefore

$$\begin{aligned} &F''(u)[B(u)M(A(u), B(u))]^{-2} \\ &\quad = \left(\log \frac{a}{b}\right)^2 \int_0^{\pi/2} f\,\mathrm{d}\mu \\ &\qquad + \left[(a-b) - A(u) \log \frac{a}{b} \right]^2 \\ &\qquad \times \left\{ 3 \int_0^{\pi/2} fg\,\mathrm{d}\mu - 2M(A(u), B(u)) \int_0^{\pi/2} f\,\mathrm{d}\mu \int_0^{\pi/2} g\,\mathrm{d}\mu \right\}. \end{aligned}$$

Using $B^2(u)f(t)+A^2(u)g(t)\equiv 1$, this becomes

$$\begin{aligned}
&F''(u)[B(u)M(A(u),B(u))]^{-2}\\
&\quad=\left(\log\frac{a}{b}\right)^2\int_0^{\pi/2} f\,\mathrm{d}\mu\\
&\qquad+\left[(a-b)-A(u)\log\frac{a}{b}\right]^2\left\{[A(u)]^{-2}\int_0^{\pi/2} f\,\mathrm{d}\mu-3\frac{B^2(u)}{A^2(u)}\int_0^{\pi/2} f^2\,\mathrm{d}\mu\right.\\
&\qquad\qquad\left.+2\frac{B^2(u)}{A^2(u)}M(A(u),B(u))\left(\int_0^{\pi/2} f\,\mathrm{d}\mu\right)^2\right\}\\
&\quad=\int_0^{\pi/2} f\,\mathrm{d}\mu\left\{\left(\log\frac{a}{b}\right)^2+\left[\frac{a-b}{A(u)}-\left(\log\frac{a}{b}\right)^2\right]^2\right\}\\
&\qquad+B^2(u)\left[\frac{a-b}{A(u)}-\log\frac{a}{b}\right]^2\left\{2M(A(u),B(u))\left(\int_0^{\pi/2} f\,\mathrm{d}\mu\right)^2-3\int_0^{\pi/2} f^2\,\mathrm{d}\mu\right\}.
\end{aligned}$$

Note that

$$F''(1)\geqslant 0\quad\text{for all choices of } 0<b<a,$$

$$F''(0)\geqslant 0\quad\text{if and only if}\quad \frac{a}{b}-1\leqslant 3\log\frac{a}{b},$$

and, since $f\leqslant[B(u)]^{-2}$,

$$\int_0^{\pi/2} f^2\,\mathrm{d}\mu\leqslant[B(u)]^{-2}\int_0^{\pi/2} f\,\mathrm{d}\mu.$$

Therefore

$$\begin{aligned}
&F''(u)[B(u)M(A(u),B(u))]^{-2}\\
&\quad\geqslant\int_0^{\pi/2} f\,\mathrm{d}\mu\left\{\left(\log\frac{a}{b}\right)^2-2\left[\frac{a-b}{A(u)}-\log\frac{a}{b}\right]^2\right\}.
\end{aligned}$$

This is nonnegative for all $0\leqslant u\leqslant 1$ if and only if

$$2^{1/2}\left[\frac{a}{b}-1-\log\frac{a}{b}\right]\leqslant\log\frac{a}{b}\quad\text{and}\quad 2^{1/2}\left[\log\frac{a}{b}-1+\frac{b}{a}\right]\leqslant\log\frac{a}{b},$$

which is equivalent to

$$\frac{a}{b}-1\leqslant\frac{1+2^{1/2}}{2^{1/2}}\log\frac{a}{b}\quad\text{and}\quad 1-\frac{b}{a}\geqslant\frac{2^{1/2}-1}{2^{1/2}}\log\frac{a}{b}.$$

This is true if $1\leqslant a/b\leqslant 2.68$. In particular, it is true if $a=2^{1/2}$, $b=1$. So, $1\leqslant a/b\leqslant 2.68$ implies that F is strictly convex on $[0,1]$. □

Theorem 4.2. *If* $1 \leq a/b \leq 2.68$, *then* $E[X(a,b)] > M(a,b)$.

Proof. If $1 \leq a/b \leq 2.68$, then, by the convexity of F proved in Lemma 4.1,

$$\int_0^1 F(u)\,\mathrm{d}u > F(\tfrac{1}{2}). \tag{4.1}$$

Now let $a = a(0)$, $b = b(0)$. Since

$$\frac{A(n)}{B(n)} \leq \frac{a}{b} \quad \text{for all } n \text{ a.s.},$$

we can apply inequality (4.1) at each stage, so

$$E\{M[A(n+1), B(n+1)] \mid A(n), B(n)\}$$
$$> M[\tfrac{1}{2}(A(n)+B(n)), [A(n)B(n)]^{1/2}] = M[A(n), B(n)].$$

Therefore

$$EM[A(n+1), B(n+1)] > EM[A(n), B(n)].$$

Since $M[A(n), B(n)] \to X(a,b)$ and $M[A(n), B(n)] \leq a$, it follows that

$$EX(a,b) = \lim_{n\to\infty} EM[A(n), B(n)] > M(a,b). \qquad \square$$

Next, we show that Conjecture 3.2 is false, at least if a/b is small.

Theorem 4.3. *For* $b = 1$, $a = 1 + t$, *where* $t > 0$ *is small*, $E[X(a,b)] < EA(2) < ED(2) < EY[(a,b)]$.

Proof. Define $f(t) = EA(2)$, $g(t) = ED(2)$. Then it is straightforward to compute that

$$f(t) = \frac{2+t}{4} + \frac{t}{2\log(1+t)}, \qquad g(t) = \int_{u=0}^{u=1} \frac{(1+tu)^u}{(1+t)^{u(u-1)}}\,\mathrm{d}u.$$

Then $f(0) = g(0) = 1$, $f'(0) = g'(0) = \frac{1}{2}$ (for the latter, differentiate under the integral and take the limit as $t \downarrow 0$), $f''(0) = g''(0) = -\frac{1}{12}$, $f'''(0) = \frac{1}{8} < g'''(0) = \frac{2}{15}$. Hence for small enough t, $f(t) < g(t)$. $\square$

Remark. For $b = 1$ and $a = 2^{1/2}$, $EA(2) = 1.201\,137\ldots$. When $t = 2^{1/2} - 1$, the above formula for $f(t)$ gives $1.201\,137\ldots$, an upper bound on the mean annealed AGM which is smaller than the sample mean for the quenched AGM given in Table 1.

The proof of Lemma 4.1 establishes that $F''(0) \geq 0$ if and only if $a/b - 1 \leq 3\log(a/b)$, i.e., if and only if $1 \leq a/b \leq 6.711$ approximately. The proof of Theorem 4.3 establishes that $E[X(a,b)] < E[Y(a,b)]$ for small values of $(a-b)/b$. To explore outside of this range, we estimated $E[X(10,1)]$ and $E[Y(10,1)]$ and the corresponding standard deviations by 1000 simulations each with convergence criterion $\delta = 10^{-10}$. We found $M(10,1) = 4.2504$, $\bar{X}(10,1) = 4.8162$, s.d.$(X) = 2.6328$, s.d.$(\bar{X}) = 0.0833$, $\bar{Y}(10, 1) = 4.8423$, s.d.$(Y) = 2.8662$, s.d.$(\bar{Y}) = 0.0906$. Thus, $\bar{X}(10, 1)$ is many times s.d.$(\bar{X})$ greater than $M(10, 1)$ and $\bar{Y}(10, 1)$ is many times s.d.$(\bar{Y})$ greater than $M(10, 1)$, supporting Conjecture 3.1. But, in conformity with Theorem 4.3, $\bar{X}(10, 1) < \bar{Y}(10, 1)$, even though the difference between these two quantities is small compared to s.d.$(\bar{X})$ and s.d.$(\bar{Y})$, in apparent but misleading conformity with Conjecture 3.2.

If Conjecture 3.1 is true, it should be provable by an argument that escapes the limitations encountered in the present proof of Lemma 4.1. For example, the inequality (4.1) might hold even if F is not convex. The following Theorem 4.4 shows that (4.1) holds for large enough a/b. Theorem 4.4 does not imply the conclusion of Theorem 4.2 when a/b is large, because $A(n)/B(n)$ may be of moderate size for some values of n. Nevertheless, Theorem 4.4 strongly suggests that (4.1), and therefore Theorem 4.2, holds for all a/b. Since Theorem 4.4 is only suggestive, we merely outline its proof.

Theorem 4.4. *For F defined just before Lemma* 4.1,

$$\lim_{a/b\to\infty} \frac{\int_0^1 F(u)\,\mathrm{d}u}{F(\frac{1}{2})} = \infty.$$

Outline of proof.

Step 1. If

$$h(a) = \int_0^{\pi/2} \frac{\mathrm{d}t}{(a^2\cos^2 t + \sin^2 t)^{1/2}} \left(= \frac{\pi}{2M(a, 1)} \right),$$

then

$$\lim_{a\to\infty} \frac{a}{\log a}\, h(a) = 1.$$

Step 2. Taking $b = 1$ and F as defined before Lemma 4.1,

$$\lim_{a\to\infty} \frac{F(u)}{F(\frac{1}{2})} = \frac{u}{1-u} \quad \text{for } 0 < u < 1.$$

To check this write

$$\frac{F(u)}{F(\frac{1}{2})} = \frac{a^u h(a)}{h((au + 1 - u)/a^u)}$$

and apply the result of Step 1.

Step 3.

$$\lim_{a\to\infty} \int_0^1 \frac{F(u)}{F(\frac{1}{2})} = \infty.$$

The proof follows from Step 2 and Fatou's lemma, since $\int_0^1 [u/(1-u)]\,\mathrm{d}u = \infty$. □

In retrospect, it is not too surprising that preliminary numerical evidence would appear to support the incorrect Conjecture 3.2. The quenched and annealed procedures agree up to the first stage, $n=1$. But most of the converging has already occurred by that stage, since if $a/b=2^{1/2}$ for example, then $EA(1)=EC(1)=1.2071\ldots$. The rest of the procedures will only change the expected values in the third decimal place. Therefore one should expect $EX-EY$ to be quite small, though not necessarily equal to zero.

Proofs or disproofs of any of the remaining conjectures would be welcome.

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