Spectral Inequalities for Matrix Exponentials

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ABSTRACT

This note generalizes an inequality of Bernstein as follows. If \( C \) is an \( n \times n \) complex matrix and \( C^{(k)} \) is the \( k \)th compound of \( C \), \( 1 \leq k \leq n \), \( N = \left( \begin{array}{c} n \\ k \end{array} \right) \), and if the eigenvalues of \( C^{(k)} \) are labeled in order of decreasing magnitude \( |\lambda_1(C^{(k)})| \geq |\lambda_2(C^{(k)})| \geq \cdots \geq |\lambda_N(C^{(k)})| \), define the partial trace \( \text{tr}^{(k)}(C) \) by

\[
\text{tr}^{(k)}(C) = \sum_{h=1}^{i} \lambda_h(C^{(k)}), \quad i = 1, \ldots, N.
\]

Then for any complex \( n \times n \) matrix \( A \),

\[
\text{tr}^{(k)}(e^{A}e^{A^*}) \leq \text{tr}^{(k)}(e^{A+A^*}), \quad i = 1, \ldots, N,
\]

with equality if \( A \) is normal or \( k = n \). A spectral inequality of K. Fan is also generalized through the use of compound matrices.

1. INTRODUCTION

Mathematical models in control theory [1], statistical mechanics [5], and population biology [2] lead to formulas containing \( e^{A}e^{B} \) and \( e^{A+B} \), for noncommuting \( n \times n \) matrices \( A \) and \( B \). The behaviors of these models depend on functions of the eigenvalues of \( e^{A}e^{B} \) and \( e^{A+B} \). The purpose of
this note is to extend a recent inequality that compares the eigenvalues of $e^{A}e^{B}$ with those of $e^{A+B}$ in the special case when $B = A^*$. 

Bernstein [1] proved, among other inequalities, that if $A$ is a real $n \times n$ matrix, $1 < n < \infty$, $A^T$ is the transpose of $A$, and $\text{tr}(A)$ is the trace of $A$, then

$$\text{tr}(e^{A}e^{A^T}) \leq \text{tr}(e^{A+A^T}).$$

(1.1)

Bernstein's proof of (1.1) relies on Theorem 3 of Fan [3, p. 654]. This note generalizes Fan's theorem and then exploits that generalization fully to extend (1.1). The remainder of this introductory section gives some notation and definitions.

As usual, for any complex $n \times n$ matrix $C$, let $C^*$ denote the conjugate transpose of $C$. A complex matrix $C$ is normal if $CC^* = C^*C$. The $k$th compound $C^{(k)}$ of $C$, for $k = 1, \ldots, n$, is the $N \times N$ matrix, where $N = \binom{n}{k}$, the elements of which are the determinants of all the possible $k \times k$ submatrices of $C$ that consist of the intersections of rows $i_1, i_2, \ldots, i_k$, where $1 \leq i_1 < \cdots < i_k \leq n$, and of columns $j_1, j_2, \ldots, j_k$, where $1 \leq j_1 < \cdots < j_k \leq n$. The elements of $C^{(k)}$ are ordered lexicographically by the indices of the rows or columns of $C$ that are included. (See [4] for a review of compound matrices.) A first key fact (e.g., [4]) is the Binet-Cauchy formula: for any complex $n \times n$ matrices $A$ and $B$, $A^{(k)}B^{(k)} = (AB)^{(k)}$, $k = 1, \ldots, n$. A second key fact is that if $\lambda_i(C)$, $i = 1, \ldots, n$, are the eigenvalues of $C$ (some of which may be repeated), then the $N$ eigenvalues of $C^{(k)}$ are all the products of eigenvalues of $C$ taken $k$ at a time:

$$\lambda_{i_1}(C)\lambda_{i_2}(C) \cdots \lambda_{i_k}(C), \quad \text{for} \quad 1 \leq i_1 < \cdots < i_k \leq n.$$ 

To illustrate, $C^{(1)} = C$ and $C^{(n)} = \det C$, where $\det = \text{determinant}$. 

Assuming the eigenvalues of $C$ are labeled in order of decreasing magnitude $|\lambda_1(C)| \geq |\lambda_2(C)| \geq \cdots \geq |\lambda_n(C)|$, define the partial trace $\text{tr}_i^{(k)}(C)$ by

$$\text{tr}_i^{(k)}(C) = \sum_{h=1}^{i} \lambda_h(C^{(k)}), \quad i = 1, \ldots, N = \binom{n}{k}.$$ 

(1.2)

Thus $\text{tr}_i^{(k)}(C) = \text{tr}_i^{(i)}(C^{(k)})$. To illustrate, $\text{tr}_i^{(k)}(C)$ is the $k$th elementary symmetric function of the eigenvalues of $C$; in particular, $\text{tr}_n^{(1)}(C)$ is the usual trace of $C$, and $\text{tr}_1^{(1)}(C)$ is the spectral radius of $C$. When $C$ is nonnegative definite, ordering the eigenvalues of $C$ by decreasing magnitude amounts to
ordering them by the usual order on nonnegative real numbers; thus $\text{tr}^{(k)}(C)$ is the product of the $k$ biggest eigenvalues of $C$.

2. INEQUALITIES FOR EXPONENTIALS OF A AND $A^*$

**Theorem 1.** For any complex $n \times n$ matrix $C$ and for any positive integer $r$,

$$\text{tr}_i^{(k)}[C'(C')^*] \leq \text{tr}_i^{(k)}[(CC^*)^r], \quad k = 1, \ldots, n, \quad i = 1, \ldots, \binom{n}{k}, \quad (2.1)$$

with equality if $C$ is normal or $k = n$.

*Proof.* The arguments of $\text{tr}_i^{(k)}(\cdot)$ in (2.1) are Hermitian nonnegative definite and therefore have real nonnegative eigenvalues, so the relation $\leq$ in (2.1) is defined.

Fan [3, p. 654] proved that for any complex $n \times n$ matrix $C$ and for any positive integer $r$,

$$\text{tr}_i^{(1)}[C'(C')^*] \leq \text{tr}_i^{(1)}[(CC^*)^r], \quad i = 1, \ldots, n. \quad (2.2)$$

Now if $C$ is replaced by $C^{(k)}$, then (by the Binet-Cauchy formula) $C^{(k)} = (C')^{(k)}$ and $(C')^{(k)*} = [(C')^*]^{(k)}$, so the argument on the left of (2.2) becomes $[C'(C')^*]^{(k)}$, and by the definition (1.2) we have $\text{tr}_i^{(1)}[(C'(C')^*])^{(k)}] = \text{tr}_i^{(k)}[C'(C')^*]$. Similarly, replacing $C$ by $C^{(k)}$ in the argument on the right of (2.2) and using the Binet-Cauchy formula give $\text{tr}_i^{(1)}[(C^{(k)}(C^{(k)*})^r] = \text{tr}_i^{(k)}[(CC^*)^r]$.

If $C$ is normal, then $C'(C')^* = (CC^*)^*$, so equality holds in (2.1). If $k = n$, both sides of (2.1) equal $(\det C)'(\det C^*)'$.

**Theorem 2.** For any complex $n \times n$ matrix $A$,

$$\text{tr}_i^{(k)}(e^Ae^{A^*}) \leq \text{tr}_i^{(k)}(e^{A+A^*}), \quad k = 1, \ldots, n, \quad i = 1, \ldots, \binom{n}{k}, \quad (2.3)$$

with equality if $A$ is normal or $k = n$.  


Proof. In (2.1), let $C = e^{A/r}$. Then, since $(e^A)^* = e^{A^*}$,

$$
tr_1^{(k)}(e^A e^{A^*}) \leq tr_1^{(k)} \left[ (e^A e^{A^*})^r \right].
$$

Let $r \uparrow \infty$ in (2.4). By the exponential product formula of Sophus Lie (e.g., [6]), $(e^A/e^{A^*})^r \to e^{A+A^*}$, which implies (2.3).

Equality holds in (2.3) when $A$ is normal because then $e^A$ is normal.

It would be interesting to know necessary and sufficient conditions for equality in (2.3).

The special case of Theorem 2 when $A$ is real, $k = 1$ and $i = n$ is (1.1) above, first proved in [1].

Dennis S. Bernstein (personal communication, 1 June 1988) points out that the square root of both sides of (2.3) in the special case $i = k = 1$ yields another known inequality: $\|e^{Ax}\| \leq e^{\mu(A)x}$, where $\| \cdot \|$ is the spectral norm (the matrix norm induced by the Euclidean vector norm), $x$ is any $n$-vector, and $\mu(A)$ is the logarithmic "norm" (also called the logarithmic derivative or the measure of a matrix). See e.g. Torsten Ström, On logarithmic norms, SIAM J. Numer. Anal. 12(5):741–753 (1975), Lemma 1c(5). Thus (2.3) unifies (1.1) with a standard inequality involving the logarithmic norm.

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