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Connectivity of finite anisotropic random graphs and directed graphs

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Abstract

For graphs on a finite set of vertices with arbitrary probabilities of independently occurring edges, the reliability is defined as the probability that the graph is connected, and the redundancy as the expected number of spanning trees of the graph. Analogous measures of connectivity are defined for random finite directed graphs with arbitrary probabilities of independently occurring directed edges. Recursive formulas for computing the reliability are known. Determinantal formulas, based on matrix-tree theorems, for computing the redundancy are given here. Among random graphs with a given sum of edge probabilities, the more evenly the probabilities are distributed over potential edges, the larger the redundancy. This inequality, proved using the theory of majorization, in combination with examples shows unexpectedly that conflicts between reliability and redundancy can arise in the design of communication networks modelled by such random graphs. The significance of these calculations for the command and control of nuclear forces is sketched.

1. *Introduction*

The purpose of this paper is to point out that in communication networks (modelled by certain anisotropic random graphs or random directed graphs), reliability and redundancy can differ markedly. If reliability is measured by the probability of connectedness and redundancy is measured by the average number of distinct ways all nodes in the network can communicate (the expected number of spanning trees), then the reliability of the network and the redundancy of the network are, in general, maximized by different assignments to individual links of probabilities of functioning, given a fixed sum of probabilities. The major new technical result in this paper is an inequality, based on majorization, that shows that, among random graphs with a given sum of edge probabilities, the expected number of spanning trees is larger, the more evenly the probabilities are distributed over potential edges.

As a possible practical application of this result, I suggest that anisotropic random graphs or digraphs form a natural family of models for the command system of the superpowers' nuclear forces. The interruption of communication between any part of the command system and its central political authority would be highly threatening both to that command system and to an opposing command system, since in such a case there would be no single locus of control or negotiation. It is therefore important to understand the reliability (probability of remaining connected) and redundancy (expected number of different ways of remaining connected) of such communication networks when communication links are randomly removed, e.g. by attack. The examples and theorems demonstrate a potential conflict between reliability and redundancy.

The connectivity of random graphs (measured by the probability that the graph is connected) seems to have been first investigated in three independent papers that appeared in the same year: Austin *et al.* [2], Erdös and Rényi [13] and Gilbert [18]. These and the further papers of Erdös and Rényi ([14], [15]; see [12] for reprints) consider only isotropic random graphs (of three different kinds). Other recent papers on the connectivity of various random graphs and random mappings include Stubbs and Good [33], Ross [31], Dorea [11] and Grimmett *et al.* [21]. Marshall ([26], chapter 7) Karoński [23] and Grimmett [20] review random graphs extensively.

I shall consider graphs ([4], [36]) and directed graphs or digraphs ([30], [36]) on the set V of vertices, $V = \{1, 2, \ldots, n\}$, $1 < n < \infty$. To model random digraphs, I shall suppose throughout that the directed edge (hereafter called *dart*, following Tutte) (i,j) from *tail* vertex i to *head* vertex j occurs with fixed but arbitrary probability $p_{ij}, 0 \leqslant p_{ij} \leqslant 1$, independently for all distinct pairs $i \neq j, 1 \leqslant i,j \leqslant n$. To model random (undirected) graphs, I shall suppose throughout that the (undirected) edge $\{i,j\}$ between i and j occurs with probability p_{ij}, independently for all distinct pairs $i < j$, $1 \leqslant i,j \leqslant n$. Such random digraphs and graphs will be called *anisotropic* to distinguish them from *isotropic* random digraphs and graphs in which necessarily $p_{ij} = p$, for all $i \neq j$.

Three analogues of a tree will be defined for digraphs. An *outtree* ([22], p. 201) from vertex i is a set of vertices and darts whose underlying graph (the graph obtained by ignoring the orientation of darts) is a tree such that i is the head of no dart in the set and every other vertex in the set is the head of exactly one dart in the set. An outtree from i is identical to Tutte's ([36], p. 126) arborescence diverging from i (but easier to say). An *intree* ([22], p. 201) to i is a digraph such that, if the orientation of every dart is reversed, the result is an outtree from i. Since the labelling of vertices is arbitrary, it entails no loss of generality when considering digraphs to deal only with outtrees from vertex 1 and intrees to vertex 1. A *bitree* is a digraph such that the underlying graph is a tree and, for every edge $\{i,j\}$ of the underlying tree, both (i,j) and (j,i) are darts of the digraph and there are no other darts.

In a digraph, a *path* is a sequence $(D_1, D_2, \ldots, D_m)$ of $m \geqslant 1$ darts D_j, not necessarily all distinct, such that the head of D_j is the tail of D_{j+1} for $1 \leqslant j < m$. The tail of D_1 and the head of D_m are called the tail and head of the path, respectively, and the path is said to go from its tail to its head. Analogous language will be used for paths of edges in graphs, except that in graphs paths have two ends (the end vertices) rather than a tail and a head.

A digraph is *strongly connected* if, for every pair i, j of vertices, there is a path from i to j. A graph is *connected* if, for every pair i, j of vertices, there is a path with ends i and j.

A tree (in-, out-, bi-, or garden-variety undirected) is *spanning* if its vertices include all of V. So a digraph with a spanning bitree is strongly connected, but a strongly connected digraph need not have a spanning bitree. However, a graph has a spanning tree if and only if it is connected.

Let $P = (p_{ij})$ be the $n \times n$ matrix of edge (or dart) probabilities for random graphs (or random digraphs). Assume throughout that P has a zero diagonal, i.e. $p_{ii} = 0$,

$i = 1, \ldots, n$. Say that random graphs are P-connected if there is a positive probability of their being connected. Similarly say that random digraphs are strongly P-connected if there is a positive probability of their being strongly connected.

A non-negative $n \times n$ matrix A is defined to be *irreducible* if, for every i and j in V, there exists a positive integer k such that $(A^k)_{ij} > 0$. Clearly, random graphs (digraphs) are (strongly) P-connected if and only if the matrix P is irreducible.

Section 2 reviews a known recursive formula for the probabilities that a random digraph has a spanning outtree from 1, that a random digraph has a spanning intree to 1, that a random digraph has a spanning bitree, and that a random graph has a spanning tree or equivalently is connected. Section 3 gives exact determinantal formulas for the expected numbers, in random digraphs, of spanning outtrees from 1, of spanning intrees to 1, and of spanning bitrees; and, for random graphs, of spanning trees. Section 4 shows that, in random graphs, the more evenly spread out a given sum of edge probabilities is, the larger the expected number of spanning trees. Examples show that the edge probability matrix that maximizes the mean number of spanning trees need not maximize the probability that a random graph has a spanning tree. If different edges or darts are associated in the sense of Esary, Proschan and Walkup [**16**], then the expected number of spanning trees (of whatever variety) is not less than the expected number of (the corresponding kind of) spanning trees under the assumption of independence. Section 5 suggests a practical interpretation of the results. Section 6 lists some open questions raised by the mathematics and by its applications.

2. *The probability of a spanning tree*

The probabilities of a connection between vertex 1 and all remaining vertices are given in Theorem 1 by a recursive formula (1) due to Kel'mans [**24**]. Let $V = \{1, 2, \ldots, n\}$ denote the set of vertices, and for i, j in V, $i \neq j$, let $V_i = V - \{i\}$, $V_{ij} = V - \{i, j\}$. For $S \subseteq V_1$, let $P_1(S)$ be the probability that vertex 1 is connected by a tree (of type to be specified) to exactly the vertices in S in a random digraph or graph on the vertex set $S \cup \{1\}$ only. Thus $P_1(V_1)$ is the probability that vertex 1 is connected to all remaining vertices in V by a tree (of type to be specified). Define H_k to be the family of all subsets of V_1 containing exactly $k-1$ elements of V_1, $1 \leqslant k \leqslant n$. E.g. $H_1 = \{\varnothing\}$, $H_n = \{V_1\}$. Also define null products (products of no factors) to equal 1.

THEOREM 1. *Let* $P_1(\varnothing) = 1$. *Then*

$$1 - P_1(V_1) = \sum_{k=1}^{n-1} \sum_{S \in H_k} P_1(S) \prod_{i \in S \cup \{1\}, j \in V_1 - S} A_{ij}. \tag{1}$$

The symbols in this recursive formula are to be interpreted as follows. In random digraphs,

(A 1) *if* $A_{ij} = q_{ij} = 1 - p_{ij}$, *then* $P_1(V_1)$ *is the probability of a spanning outtree from* 1;

(A 2) *if* $A_{ij} = q_{ji}$, *then* $P_1(V_1)$ *is the probability of a spanning intree to* 1;

(A 3) *if* $A_{ij} = q_{ij} + q_{ji} - q_{ij}\, q_{ji}$, *then* $P_1(V_1)$ *is the probability of a spanning bitree.*

In random graphs,

(A 4) *if* $A_{ij} = q_{ij}$, *then* $P_1(V_1)$ *is the probability of a spanning tree.*

Kel'mans ([**24**] equation (3)) and independently Buzacott ([7], p. 314) derive (1) for graphs. Buzacott [8] extends (1) to outtrees and intrees. The extension to bitrees is trivial. I illustrate (1) for spanning outtrees of random digraphs and for spanning trees of random graphs. When $q_{ij} = q_{ji}$ for all i, j, these formulas reduce to the examples given by Kel'mans ([**24**], appendix 1). When $q_{ij} = q$ for all $i \neq j$, these equations reduce to the illustrative equations Gilbert ([**18**], p. 1142) gives for isotropic random graphs:

$$n = 2:\ 1 - P_1(\{2\}) = q_{12}.$$

$$\begin{aligned} n = 3:\ 1 - P_1(\{2,3\}) &= q_{12}q_{13} + P_1(\{2\})q_{13}q_{23} + P_1(\{3\})q_{12}q_{32} \\ &= q_{12}q_{13} + q_{13}q_{23} + q_{12}q_{32} - q_{12}q_{13}q_{23} - q_{13}q_{12}q_{32}; \end{aligned}$$

$$\begin{aligned} n = 4:\ 1 - P_1(\{2,3,4\}) &= q_{12}q_{13}q_{14} + P_1(\{2\})q_{13}q_{14}q_{23}q_{24} \\ &\quad + P_1(\{3\})q_{12}q_{14}q_{32}q_{34} + P_1(\{4\})q_{12}q_{13}q_{42}q_{43} \\ &\quad + P_1(\{2,3\})q_{14}q_{24}q_{34} + P_1(\{2,4\})q_{13}q_{23}q_{43} + P_1(\{3,4\})q_{12}q_{32}q_{42} \\ &= q_{12}q_{13}q_{14} + q_{14}q_{24}q_{34} + q_{13}q_{23}q_{43} + q_{12}q_{32}q_{42} \\ &\quad + q_{13}q_{14}q_{23}q_{24} + q_{12}q_{14}q_{32}q_{34} + q_{12}q_{13}q_{42}q_{43} \\ &\quad - q_{12}q_{13}q_{14}q_{23}q_{24} - q_{13}q_{12}q_{14}q_{32}q_{34} - q_{14}q_{12}q_{13}q_{42}q_{43} \\ &\quad - q_{14}q_{24}q_{34}(q_{12}q_{13} + q_{13}q_{23} + q_{12}q_{32}) \\ &\quad - q_{13}q_{23}q_{43}(q_{12}q_{14} + q_{14}q_{24} + q_{12}q_{42}) \\ &\quad - q_{12}q_{32}q_{42}(q_{14}q_{13} + q_{13}q_{43} + q_{14}q_{34}) \\ &\quad + q_{14}q_{24}q_{34}(q_{12}q_{13}q_{23} + q_{13}q_{12}q_{32}) \\ &\quad + q_{13}q_{23}q_{43}(q_{12}q_{14}q_{24} + q_{14}q_{12}q_{42}) \\ &\quad + q_{12}q_{32}q_{42}(q_{14}q_{13}q_{43} + q_{13}q_{14}q_{34}). \end{aligned}$$

$P_1(V_1)$ involves none of q_{i1}, $1 < i \leqslant n$, but does depend on all $n^2 - 2n + 1$ remaining off-diagonal elements of the matrix (q_{ij}).

Similar recursive formulas for the probability that there is a path with tail vertex 1 and head vertex 2 in a random digraph on $n > 1$ vertices, and the probability that there is a path with end vertices 1 and 2 in a random graph, are given by Buzacott ([**7**], pp. 321–322; [**8**], pp. 244–245) and Provan and Ball ([**29**], p. 520).

3. *The expected number of connections*

For any $n \times n$ real matrix $A = (a_{ij})$, following the terminology of Tutte ([**36**], p. 138), define the $n \times n$ Kirchhoff matrix $K(A)$ of A by

$$K_{ii} = \sum_{j \neq i} a_{ij} \quad (i = 1, \ldots, n),$$

$$K_{ij} = -a_{ij} \quad (1 \leqslant i \neq j \leqslant n).$$

$K(A)$ depends only on the off-diagonal elements of A. K is singular since all its row sums are 0. If B is also a real $n \times n$ matrix and a and b are real scalars, then

$$K(aA + bB) = aK(A) + bK(B).$$

Also $K(K(K(A))) = K(A)$.

For any $n \times n$ matrix A, let $\det A$ be the determinant of A and let $A(i_1, \ldots, i_q)$ be the $(n-q) \times (n-q)$ principal submatrix of A formed by striking out rows and columns $i_1, \ldots, i_q$, for $1 \leqslant q \leqslant n$. If $q = 0$, define $A(\varnothing) = A$. If $\{i_1, \ldots, i_q\} = Q \subseteq V$, abbreviate $A(Q) = A(i_1, \ldots, i_q)$.

THEOREM 2. *The expected number $E(T)$ of spanning trees (of a type to be specified) is given by*

$$E(T) = \det K(A)(1),$$

i.e. the determinant of the matrix formed from the Kirchhoff matrix $K(A)$ of A by striking out the first row and column of $K(A)$ [not the Kirchhoff matrix of A after the first row and column of A have been been struck out] when A is interpreted as follows. In random digraphs,

(B 1) *if $a_{ij} = p_{ij}$ (the probability of a dart from i to j), then $E(T)$ is the expected number of spanning intrees to* 1;

(B 2) *if $a_{ij} = p_{ji}$, then $E(T)$ is the expected number of spanning outtrees from* 1;

(B 3) *if $a_{ij} = p_{ij}p_{ji}$, then $E(T)$ is the expected number of spanning bitrees.*

In random graphs,

(B 4) *if $a_{ij} = p_{ij}$ (the probability of an edge between i and j), then $E(T)$ is the expected number of spanning trees.*

Proof. (B 1) If Δ is any spanning intree to 1, then the probability that Δ is contained as a subdigraph of a random digraph is the product $\Pi(\Delta)$ of p_{ij} over all darts (i,j) of Δ. Then the expected number $E(T)$ of spanning intrees to 1 is

$$E(T) = \sum_{\Delta} \Pi(\Delta), \tag{2}$$

where the sum is over all spanning intrees to 1. The matrix-tree theorem for digraphs of Tutte ([**36**], p. 140) asserts that the sum on the right equals $\det K(P)(1)$, where P is the $n \times n$ matrix of dart probabilities.

(B 2) follows from (B 1) upon reversing the orientation of darts and replacing p_{ij} by p_{ji}. The proof of (B 4) parallels that of (B 1), using the matrix-tree theorem for graphs (Brooks *et al.* [**6**] (who suggest that their theorem is due 'in principle' to Kirchhoff in 1847 and Borchardt in 1860), Trent [**35**], Tutte ([**36**], p. 141)). (B 3) is immediate from (B 4). ∎

An alternative probabilistic proof of Theorem 2 (B 4) (I omit the elementary but lengthy details) uses induction on n and decomposes $E(T)$ into a sum of conditional expectations. Instead of the matrix-tree theorem for nonrandom graphs, this proof uses the following interesting expansion of the determinant (e.g. Aitken [**1**], pp. 87–88):

PROPOSITION 1. *Let A be an $n \times n$ matrix, X an $n \times n$ diagonal matrix,* i.e. $x_{ij} = 0$ *if* $i \neq j$, $x_{ii} = x_i$. *Let $A(V)$ denote the matrix A with all of its rows and columns struck out; let* $\det A(V) = 1$ *and* $\prod_{j \in \varnothing} x_j = 1$ *(i.e. let the determinant of the null matrix and the product of no factors equal* 1); *let $\#(Q)$ denote the number of vertices in the subset Q of V. Then*

$$\det(A+X) = \sum_{q=0}^{n} \sum_{Q \subseteq V,\, \#(Q)=q} (\det A(Q)) \prod_{j \in Q} x_j.$$

For example, if $n = 2$, $\det(A+X) = x_1x_2 + a_{11}x_2 + a_{22}x_1 + \det A$.

The speculation that there might be a determinantal formula for $P_1(V_1)$ analogous to that for $E(T)$ is destroyed by the demonstration of Provan and Ball [**28**] that the two quantities are of different computational complexity.

COROLLARY 2·1. *The expected number of spanning trees in a random graph when all* *vertices in the subset H of the vertex set V have been collapsed to a single vertex is* $E(T; n, H) = \det K(P)(H)$.

The preceding corollary generalizes the analogous result for multigraphs (e.g. [**10**], p. 38).

COROLLARY 2·2. *If $\mu_1, \ldots, \mu_n$ are the (nonrandom) eigenvalues, repeated according to their multiplicity, of the (symmetric) $n \times n$ Kirchhoff matrix $K(P)$ of the edge probability matrix P, then they are real and non-negative and, when ordered $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n = 0$, satisfy*

$$E(T) = n^{-1} \prod_{i=1}^{n-1} \mu_i.$$

Thus $E(T) > 0$ if and only if $\mu_{n-1} > 0$.

Proof. It follows readily from theorem 3 of Trent ([35], p. 1007) that $K(P)$ is non-negative definite, so $\mu_i \geqslant 0, i = 1, \ldots, n$. Also $\mu_n = 0$ because $\det K(P) = \prod_{i=1}^n \mu_i = 0$. Since $E(T) = \det K(P)(i)$, $i = 1, \ldots, n$, $E(T) = n^{-1} \sum_{i=1}^n \det K(P)(i)$. By the theory of elementary symmetric functions of matrix spectra,

$$\sum_{i=1}^{n} \det K(P)(i) = \sum_{i=1}^{n} \prod_{j \neq i} \mu_j = \prod_{j \neq n} \mu_j$$

since those products that contain the factor μ_n vanish. Hence $E(T) = n^{-1} \prod_{i=1}^{n-1} \mu_i$. Hence $E(T) = 0$ if and only if $\mu_{n-1} = 0$. ∎

Cvetković *et al.* ([**10**], p. 39) give a formula analogous to that of Corollary 2·2 for the number of spanning trees of a nonrandom connected multigraph.

For each fixed graph G on n vertices, let K_G be its Kirchhoff matrix, i.e. $(K_G)_{ii}$ is the number of vertices in V to which i is adjacent and for $i \neq j$ $(K_G)_{ij} = -1$ if $\{i, j\}$ is an edge of G, $(K_G)_{ij} = 0$ if not. Clearly $E(K_G) = K(P)$.

COROLLARY 2·3. *For i in V,* $\det K(P)(i) = E(\det K_G(i)) = E(T)$.

Proof. $\det K_G(i)$ is the number of spanning trees of G by the matrix-tree theorem for ordinary graphs, so $E(\det K_G(i)) = E(T)$. ∎

The probability that a random graph G is *not* connected, which is given recursively by Theorem 1, equals the probability that $\det K_G(i)$ equals 0, for any i in V, because G is not connected if and only if G has no spanning trees.

COROLLARY 2·4. *In an isotropic random graph (or digraph) on n vertices, the expected number of spanning trees (or outtrees or intrees to any vertex) is $E(T) = p^{n-1}n^{n-2}$. For an isotropic random digraph, the expected number of spanning bitrees is $p^{2(n-1)}n^{n-2}$.*

This easy fact is given, for graphs, by Grimmett ([**19**], p. 118).

Let J be the $n \times n$ matrix with all elements 1.

COROLLARY 2·5. *For random graphs on n vertices with matrix P of edge probabilities, the expected number of spanning trees is given by*

$$E(T) = n^{-2} \det (J + K(P)).$$

This corollary is a direct extension of a formula due to Temperley [**34**]. The proof repeats the proof of Biggs ([3], p. 35) step by step. The formula does not apply to spanning out- or intrees of random digraphs. The formula can be modified to apply to bitrees of random digraphs by replacing the matrix P with the matrix with (i,j) element $p_{ij} p_{ji}$.

COROLLARY 2·6. *For random graphs with probability matrix P, the following statements are equivalent:*

(i) *P is irreducible.*
(ii) *The random graphs are P-connected.*
(iii) *The probability of a spanning tree is positive, i.e. $P_1(V_1) > 0$.*
(iv) *The expected number of spanning trees is positive, i.e. $E(T) > 0$.*
(v) $\det K(P)(i) > 0, i = 1, \ldots, n$.
(vi) *The next-to-smallest eigenvalue μ_{n-1} of $K(P)$ is positive.*

4. *Inequalities*

The main object of this section is a majorization inequality in Theorem 3 that compares the expected number of spanning trees in anisotropic random graphs with the expected number of spanning trees in isotropic random graphs with the same total of edge probabilities.

If x and y are two real n-vectors, $x = (x_1, \ldots, x_n)^T$, $y = (y_1, \ldots, y_n)^T$, let $x_{[1]} \geqslant \ldots \geqslant x_{[n]}$ denote the elements of x in decreasing order, and similarly for y. Following Marshall and Olkin [27], say that x is majorized by y and write

$$x \prec y \quad \text{if} \quad \sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]} \quad (k = 1, \ldots, n-1)$$

$$\text{and} \quad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}.$$

LEMMA 3·1. *Let x be a positive n-vector ($x_i > 0$, $i = 1, \ldots, n$) and let $\bar{\mathbf{x}}$ be the n-vector with all elements equal to $\bar{x} = n^{-1} \sum_{i=1}^{n} x_i$. Let $x(\alpha) = (1-\alpha)x + \alpha\bar{\mathbf{x}}$, for $0 \leqslant \alpha \leqslant 1$. Then $x(\alpha_1) \prec x(\alpha_2)$ if $\alpha_1 \geqslant \alpha_2$.*

Proof. If $\alpha_1 \geqslant \alpha_2$, then, for $k = 1, \ldots, n-1$,

$$\begin{aligned}\sum_{i=1}^{k} x(\alpha_1)_{[i]} &= (1-\alpha_1) \sum_{i=1}^{k} x_{[i]} + \alpha_1 k\bar{x} \\ &= \sum_{i=1}^{k} x_{[i]} - \alpha_1 \sum_{i=1}^{k} (x_{[i]} - \bar{x}) \leqslant \sum_{i=1}^{k} x_{[i]} - \alpha_2 \sum_{i=1}^{k} (x_{[i]} - \bar{x}) \\ &= \sum_{i=1}^{k} x(\alpha_2)_{[i]}\end{aligned}$$

and obviously
$$\sum_{1}^{n} x(\alpha_1)_{[i]} = \sum_{1}^{n} x(\alpha_2)_{[i]}. \quad \blacksquare$$

A similar result is given by Marshall and Olkin ([27], p. 130, B.2.a).

LEMMA 3·2 ([27], p. 79, F.1.a.) *Let x and y be positive n-vectors such that $x \prec y$. Then $\prod_1^n x_i \geqslant \prod_1^n y_i$ with strict inequality unless x is a permutation of y.*

For strict inequality, it is not sufficient to assume merely that x and y are non-negative. E.g. if $x = (0, 1, 1)^T$ and $y = (0, \frac{1}{2}, \frac{3}{2})^T$, then $x \prec y$ but $\prod x_i = \prod y_i = 0$.

For any $n \times n$ real matrix A with zero diagonal, define the $n \times n$ real matrix $\bar{A} = (\bar{a}_{ij})$ to be the *equisummed matrix of A* if $\bar{a}_{ii} = 0$, $i = 1, \ldots, n$, and,

$$\text{for all } i \neq j,\ 1 \leqslant i,j \leqslant n, \quad \bar{a}_{ij} = \sum_{g,h} a_{gh}/[n(n-1)] \equiv p(A).$$

All the off-diagonal elements of $\bar{A}$ equal the average $p(A)$ of the off-diagonal elements of A.

LEMMA 3·3. *Let A be a symmetric irreducible real $n \times n$ matrix with non-negative off-diagonal elements and zero diagonal elements, and let $\bar{A}$ be the equisummed matrix of A. If the eigenvalues of $K(A)$ are $\mu_1 \geqslant \ldots \geqslant \mu_{n-1} > \mu_n = 0$, then the eigenvalues of*

$$K_\alpha = (1-\alpha)\,K(A) + \alpha K(\bar{A})$$

are $\mu_h(\alpha) = (1-\alpha)\,\mu_h + \alpha p(A)\,n$, for $h = 1, \ldots, n-1$ and $\mu_n(\alpha) = 0$, for $0 \leqslant \alpha \leqslant 1$.

Proof. The assumption that A is irreducible implies $\mu_h > 0$, $h = 1, \ldots, n-1$, by Corollary 2·6 (possibly after rescaling A). Let u be an eigenvector of $K(A)$ corresponding to a positive eigenvalue μ. Then $K(A)\,u = \mu u$ implies $\mathbf{1}^T K(A)\,u = \mathbf{0}^T u = 0 = \mu \mathbf{1}^T u$, hence $0 = \mathbf{1}^T u$, where $\mathbf{1}$ is the n-vector with all elements 1. Therefore $Ju = \mathbf{0}$. Then $K_\alpha u = (1-\alpha)\,K(A)\,u + \alpha p(A)\,[nI - J]\,u = (1-\alpha)\,\mu u + \alpha p(A)\,nu$, which implies that $(1-\alpha)\,\mu + \alpha p(A)\,n$ is an eigenvalue of K_α corresponding to the eigenvector u. Since $K_\alpha \mathbf{1} = \mathbf{0}$, $\mu_n(\alpha) = 0$, for $0 \leqslant \alpha \leqslant 1$. ∎

We now drop the assumption in Lemma 3·3 that A is irreducible.

THEOREM 3. *Let A be a symmetric real $n \times n$ matrix with non-negative off-diagonal elements and zero diagonal elements. Let $\bar{A}$ be the equisummed matrix of A. Let*

$$K_\alpha = (1-\alpha)\,K(A) + \alpha K(\bar{A}),$$

for $0 \leqslant \alpha \leqslant 1$. If $A \neq \bar{A}$, then $\det K_\alpha(i)$ is an increasing (i.e. strictly increasing) function of α on $[0, 1]$. ($K_\alpha(i)$ is the $(n-1) \times (n-1)$ matrix formed from K_α by deleting row and column i.)

Proof. Since $\det K_\alpha(i)$ is a continuous function of α on $[0, 1]$, it suffices to prove that $\det K_\alpha(i)$ increases with α on $(0, 1)$. Therefore pick $0 < \alpha_1 < \alpha_2 \leqslant 1$.

The hypothesis $A \neq \bar{A}$ implies that $A \neq 0$, which in turn implies that $\bar{A}$ and $A_\alpha \equiv (1-\alpha)\,A + \alpha\bar{A}$ are irreducible for any α in $(0, 1]$. So for α in $(0, 1]$, A_α satisfies the hypotheses of Lemma 3·3.

Applying Lemma 3·1 to the $(n-1)$-vector $(\mu_1(\alpha), \ldots, \mu_{n-1}(\alpha))^T = \boldsymbol{\mu}(\alpha)$ of positive eigenvalues of K_α given explicitly in Lemma 3·3 shows that $\boldsymbol{\mu}(\alpha_2) \prec \boldsymbol{\mu}(\alpha_1)$. By Lemma 3·2,

$$\prod_{h=1}^{n-1} \mu_h(\alpha_1) < \prod_{h=1}^{n-1} \mu_h(\alpha_2),$$

since, if $A \neq \bar{A}$, $\boldsymbol{\mu}(\alpha_1)$ is not a permutation of $\boldsymbol{\mu}(\alpha_2)$ by Lemma 3·3. Therefore

$$\det K_\alpha(i) = n^{-1} \prod_{h=1}^{n-1} \mu_h(\alpha)$$

increases strictly with α. ∎

For any $n \times n$ matrix A, let $\operatorname{sp} A$ denote the spectrum of A, i.e. the set of eigenvalues of A, each repeated according to its multiplicity.

LEMMA 3·4. *Let A be a symmetric real $n \times n$ matrix with non-negative off-diagonal elements and zero diagonal elements, and let $\bar{A}$ be the equisummed matrix of A. Then $\operatorname{sp} K(A) = \operatorname{sp} K(\bar{A})$ if and only if $A = \bar{A}$.*

Proof. Assume first that $p(A) = 0$. Since A is elementwise non-negative, $p(A) = 0$ if and only if $A = \mathbf{0}$. Therefore $A = \bar{A} = K(A) = K(\bar{A}) = \mathbf{0}$ so $\operatorname{sp} K(A) = \operatorname{sp} K(\bar{A}) = 0$. The converse is equally obvious.

Henceforth assume $p(A) > 0$. If an arbitrary $n \times n$ matrix L has k *distinct* eigenvalues $\lambda_1, \ldots, \lambda_k$, $1 \leqslant k \leqslant n$, each with multiplicity $m_1, \ldots, m_k$, $m_1 + \ldots + m_k = n$, $m_h \geqslant 1$, $h = 1, \ldots, k$, then let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$, $\mathbf{m} = (m_1, \ldots, m_k)$ and write $\operatorname{sp}(L) = (\boldsymbol{\lambda}; \mathbf{m})$. With this notation, it is easy to calculate $\operatorname{sp} K(\bar{A})$ explicitly. Since J is positive and of rank one with all row sums equal to n, $\operatorname{sp}(J) = ((n, 0);\ (1, n-1))$. (This means that the eigenvalues of J are n, with multiplicity one, and 0 with multiplicity $n-1$.) Therefore

$$\operatorname{sp}(-J) = ((0, -n);\ (n-1, 1)) \quad \text{and} \quad \operatorname{sp}(nI - J) = ((n, 0);\ (n-1, 1)).$$

With $p(A) = \sum_{g,h} a_{gh}/[n(n-1)]$, $K(\bar{A}) = p(A)(nI - J)$. So

$$\operatorname{sp} K(\bar{A}) = ((p(A)\, n, 0);\ (n-1, 1)).$$

Let the eigenvalues of $K(A)$ be $\mu_1 \geqslant \ldots \geqslant \mu_{n-1} \geqslant \mu_n = 0$. If $A = \bar{A}$, then $\mu_h = p(A)n$ for $h = 1, \ldots, n-1$, so $\operatorname{sp} K(A) = \operatorname{sp} K(\bar{A})$.

Conversely, suppose $\operatorname{sp} K(A) = \operatorname{sp} K(\bar{A})$. Then $\mu_h = p(A)n$ for $h = 1, \ldots, n-1$ and $\mu_n = 0$ so $\operatorname{sp} K(A) = ((p(A)\, n, 0);\ (n-1), 1))$ and $\operatorname{sp}[p(A)^{-1} K(A)] = ((n, 0);\ (n-1, 1))$ and $\operatorname{sp}[-p(A)^{-1} K(A)] = ((0, -n);\ (1, n-1))$ and

$$\operatorname{sp}[nI - p(A)^{-1} K(A)] = ((n, 0);\ (1, n-1)).$$

Let $X = nI - p(A)^{-1} K(A)$. Since $\operatorname{sp} X = ((n, 0);\ (1, n-1))$, X is of rank 1, i.e. there exist n-vectors u, v such that $X = uv^T$, where v^T is the transpose of v. Since A is symmetric, so are $K(A)$ and X, and hence $u = v$, i.e. $X = uu^T$. If x is the right eigenvector of X corresponding to the eigenvalue n, then $Xx = u(u^T x) = nx$ implies $u = x$ and $u^T x = u^T u = \sum u_i^2 = n$. Then

$$\mathbf{1}^T X \mathbf{1} = (\mathbf{1}^T u)^2 = \mathbf{1}^T [nI - p(A)^{-1} K(A)]\, \mathbf{1} = n \mathbf{1}^T I \mathbf{1} = n^2,$$

so $\pm \mathbf{1}^T u = \pm \sum u_i = n$. But $\pm \sum u_i = \sum u_i^2 = n$ imply $u_i = \pm 1$, i.e. $X = J$. Hence $K(A) = p(A)(nI - J) = K(\bar{A})$, so $A = \bar{A}$. ∎

COROLLARY 3·1. *Let A be a symmetric real $n \times n$ matrix with non-negative off-diagonal elements and zero diagonal elements, and let $\bar{A}$ be the equisummed matrix of A. Then for $1 \leqslant i, j \leqslant n$,*

$$\det K(A)(i) \leqslant \det K(\bar{A})(j) = [p(A)]^{n-1} n^{n-2},$$

with equality if and only if $A = \bar{A}$. [$K(\bar{A})$ (j) denotes the $(n-1) \times (n-1)$ matrix formed by striking out row and column j of the Kirchhoff matrix $K(\bar{A})$ of $\bar{A}$, and similarly for $K(A)$ (i).]

Proof. The inequality follows from Theorem 3 and the necessary and sufficient condition for equality follows from Lemma 3·4. ∎

In the notation of Tutte ([**36**], p. 141), let Δ be any spanning tree of an electrical network on n vertices and let $\Pi(\Delta)$ be the product of the conductivities of the edges of Δ. Let $\sum \Pi(\Delta)$ denote the sum of $\Pi(\Delta)$ over all spanning trees Δ of the network.

COROLLARY 3·2. *If p is the average conductivity of the network, then the maximum of $\sum \Pi(\Delta)$ over all networks with average conductivity p is $p^{n-1}n^{n-2}$ and this maximum is attained if and only if all conductivities equal the average p.*

Proof. Let a_{ij} be the conductivity between vertices i and j $(i \neq j)$; apply Theorem 3 to the matrix-tree theorem of Tutte ([**36**], p. 141) which gives

$$\Sigma \Pi (\Delta) = \det K(A)(i) \quad (i = 1, \ldots, n). \quad |$$

COROLLARY 3·3. *If the matrix $P = (p_{ij})$ of edge probabilities (symmetric, zero diagonal, and non-negative) is anisotropic (i.e. for some $i \neq j$, $p_{ij} \neq p(P) \equiv \sum_{i,j:i \neq j} p_{ij}/[n(n-1)]$), then the expected number $E_\alpha(T)$ of spanning trees of random graphs with edge probabilities $P_\alpha = (1-\alpha)P + \alpha\bar{P}$ (where $\bar{P}$ is the equisummed matrix of P, $\bar{P} = p(P)(J-I)$) increases strictly with α. The maximum of $E(T)$ over all P with $\sum_{i,j} p_{ij} = p(P)\,n(n-1)$ for some* fixed *value of $p(P)$, $0 \leqslant p(P) \leqslant 1$, is $p(P)^{n-1}n^{n-2}$. This maximum is attained if and only if $P = p(P)(J-I)$, i.e. if and only if the random graphs are isotropic.*

Proof. Letting $A = P$, apply Theorem 3 and Theorem 2 (B 4). |

The conclusions of Theorem 3 and Corollary 3·3 may be strengthened to assert that $\det K_\alpha(i)$ and $E_\alpha(T)$, respectively, are also strictly concave functions of α on $[0, 1]$ when $n = 3$, but not in general if $n > 3$.

Neither of the requirements in Theorem 3 that A be symmetric and non-negative (off the main diagonal) can be relaxed. Hence Corollary 3·3 does not generalize to the expected number of spanning outtrees or intrees of anisotropic random digraphs. For example, if $n = 2$, the expected number of spanning outtrees from vertex 1 is p_{12}. Since $p = (p_{12}+p_{21})/2$, the expected number p of spanning outtrees from vertex 1 in isotropic random digraphs will be smaller, contrary to the conclusion of Corollary 3·3, if $p_{21} < p_{12}$.

When P is not required to be symmetric, $\det K(P)(1)$ is entirely independent of p_{1j}, $j = 2, \ldots, n$, though not of p_{i1}, $i = 2, \ldots, n$, because the latter affect the diagonal elements of $K(P)$ even though they do not appear directly in $K(P)(1)$. It might seem more natural therefore to choose the probability p for isotropic random digraphs (to compare with anisotropic random digraphs that have a given P) without reference to the first row of P. However, even if we require only that

$$\sum_{i=2}^{n} \sum_{j=1}^{n} p_{ij} = p(n-1)^2$$

(recalling that $p_{ii} = 0, i = 1, \ldots, n$), the inequality asserted by Corollary 3·3 for graphs may fail for digraphs. For example, if $n = 3$, let $p_{21} = 0{\cdot}2, p_{23} = 0{\cdot}1, p_{31} = 0{\cdot}1, p_{32} = 0{\cdot}2$ (symmetry is violated because $p_{23} \neq p_{32}$). Then $p = 0{\cdot}15$ and

$$\det[p(3I-J)](1) = (0{\cdot}3)^2 - (0{\cdot}15)^2 < \det K(P)(1) = (0{\cdot}3)^2 - (0{\cdot}1)(0{\cdot}2).$$

In Theorem 3, if A is permitted to have negative off-diagonal elements but required to satisfy all the other given conditions, the desired inequality may fail. For example, if $n = 4$, let A have zero diagonal and all off-diagonal elements equal to -1, except $a_{41} = a_{14} = -2$. Hence $p = -14/12$. Then

$$\det K(A)(1) = -24 > n^{n-2}p^{n-1} = -16(14/12)^3 = -25{\cdot}407.$$

Other examples show that, when A is not required to be non-negative, it can happen that $|p^{n-1}n^{n-2}| < |\det K(A)(1)|$.

Corollary 3·3 says that, for a fixed sum of probabilities of edges, the more evenly spread out the probabilities are, the greater the expected number of spanning trees. The expected number of spanning trees of random graphs is maximized when, and only when, the fixed sum of probabilities is evenly, i.e. isotropically, distributed.

A few examples will now show that the behaviour of the probability $P_1(V_1)$ that a random graph is connected is not nearly so simple as that of $E(T)$. To make explicit the dependence on the matrix P, let $C(P) \equiv P_1(V_1)$ be the probability that random graphs with probability matrix P are connected. Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then $C(A) = 0 < C(\bar{A}) = 7/27$, while $C(B) = 1 > C(\bar{B}) = 20/27$. (To compute e.g. $C(\bar{A})$, one finds $p = \sum a_{ij}/[n(n-1)] = \frac{1}{3}$, $q = \frac{2}{3}$ and, using Gilbert's formula,

$$C(\bar{A}) = 1 - 3q^2 + 2q^3 = 7/27.)$$

Among edge-probability matrices P with $\sum_{i,j} p_{ij} \geqslant 2(n-1)$, there is obviously at least one, say P^*, that guarantees $C(P^*) = 1$, namely by assigning $p^*_{i1} = p^*_{1i} = 1$, for $i = 2, \ldots, n$, and distributing the balance (if any) of $\sum_{i,j} p_{ij} - 2(n-1)$ arbitrarily among the remaining possible edges. According to P^*, there is an edge between vertex 1 and each other vertex with probability 1; hence $C(P^*) = 1$.

Among edge-probability matrices P with $\sum_{i,j} p_{ij} \equiv pn(n-1) < 2(n-1)$, it appears reasonable to conjecture that $C(P)$ would be maximized by P^* with

$$p_{i1} = p_{1i} = pn/2 \quad (i = 2, \ldots, n),$$

for which $C(P^*) = (pn/2)^{n-1}$. Unfortunately this conjecture is false. Let

$$P = \begin{pmatrix} 0 & 0{\cdot}1 & 0{\cdot}3 \\ 0{\cdot}1 & 0 & 0{\cdot}2 \\ 0{\cdot}3 & 0{\cdot}2 & 0 \end{pmatrix}.$$

By Theorem 1, $C(P) = 0{\cdot}098 > C(P^*) = (0{\cdot}3)^2 = 0{\cdot}09$.

Section 5 suggests an interpretation of the preceding inequalities. The balance of this section digresses to mention some related inequalities.

Bounds for $C(P) = P_1(V_1)$ follow easily from the bounds on $C(P)$ of Gilbert ([**18**], p. 1143) for isotropic random graphs. Kel'mans [**24**] derives bounds for $P_1(V_1)$ for anisotropic random graphs independently of Gilbert [**18**]. In addition, it is pretty obvious, though worth stating until an earlier source for the statement can be found, that

$$P_1(V_1) \leqslant E(T),$$

with the same inequality for the probabilities and expected numbers of spanning outtrees, intrees and bitrees of random digraphs. Bonferroni's inequalities (e.g. [**17**], p. 110) imply both the inequality and conditions for equality. Kel'mans [**25**] gives more sophisticated relations between the probabilities of connectedness and the Kirchhoff matrix of anisotropic random graphs.

For many applications, the assumption that darts or edges are present or absent independently of one another may not be satisfactory. Following Esary, Proschan and Walkup[**16**], define the random variables $X_1, \ldots, X_N$ to be *associated* if, for

$$X = (X_1, \ldots, X_N)^T, \quad \mathrm{Cov}\,[f(X), g(X)] \geqslant 0$$

for all nondecreasing functions f and g for which $Ef(X)$, $Eg(X)$ and $Ef(X)g(X)$ exist. If $\{X_i\}$ are associated binary random variables, i.e. taking only the values 0 and 1, then

$$P(X_1 = 1, \ldots, X_N = 1) \geqslant P(X_1 = 1) \ldots P(X_N = 1). \tag{3}$$

The darts or edges of a random digraph or random graph are said to be associated if each dart or edge is represented by a binary random variable (equal to 0 when the dart or edge is absent and equal to 1 otherwise) and the collection of binary random variables is associated.

PROPOSITION 2. *Let $E_0(T)$ be the expected number of spanning out-, in-, or bitrees in a random digraph or the expected number of spanning trees in a random graph, when darts or edges occur independently with probabilities given by the matrix P; $E_0(T)$ is identical to $E(T)$ in Theorem 2. Let $E_+(T)$ be the expected number of spanning trees (of corresponding type) in a random digraph or random graph when darts or edges are associated and have* marginal *probabilities P of occurring; p_{ij} is the marginal probability of a dart or edge from vertex i to vertex j. Then*

$$E_+(T) \geqslant E_0(T).$$

Proof. Let Δ be any spanning outtree from 1, $P(\Delta)$ be the probability that Δ is contained as a subdigraph of a random digraph in which darts are associated, and $\Pi(\Delta)$ denote the product of p_{ij} over all darts (i,j) of Δ. Then, using (2) and (3),

$$E_+(T) = \sum P(\Delta) \geqslant \sum \prod(\Delta) = E_0(T).$$

The proof for the other cases is the same. ∎

A concept of negatively associated random variables that implied the reverse of the inequality (3) would imply the reverse of the inequality in Proposition 2. The referee suggests that the random graphs of Erdös and Rényi[**14**] with a fixed number of edges may be candidates to consider here.

Once again, the behaviour of $C(P)$, the probability that a random graph is connected given edge probability matrix P, is more complicated than that of $E(T)$. Consider a random graph on three vertices in which all edges are perfectly associated, i.e. with probability 1, if any one edge occurs, then all edges occur, and, if any one edge fails to occur, then no edges occur. If all edges occur with marginal probability p in $[0, 1]$, then $C(P) = p$. By contrast, for an isotropic random graph with independent edges, it is easy to see (from the formula for $n = 3$ following Theorem 1 or from the formula of Gilbert[**18**]) that the probability of a spanning tree is less than p for p in $(0, \frac{1}{2})$, is greater than p for p in $(\frac{1}{2}, 1)$, and equals p for $p = 0, \frac{1}{2}, 1$. Thus a general inequality for $C(P)$ analogous to that of Proposition 2 for $E(T)$ does not hold.

5. *Command and control of distributed forces*

Bracken ([**5**], p. 124) gives a stylized representation of the command system of the United States' strategic forces as a graph on six vertices, labelled EUR (European Command), LANT (Atlantic Command), PAC (Pacific Command), SAC (Strategic Air Command), NCA (National Command Authorities, the political command including the President and the Joint Chiefs of Staff), and IONDS (Integrated Operational Nuclear Detonation Detection System, a damage-assessment system based on satellites). In greater detail, according to Steinbruner ([**32**], p. 38), 'the U.S. has dispersed the physical ability to fire nuclear weapons among hundreds of military officers at numerous locations, some of them mobile. From all public indications the U.S.S.R. has done the same'.

Bracken writes ([**5**], pp. 122–123), 'Any time a breakdown in communication channels is severe enough that individual commanders are essentially operating on their own with little knowledge of the overall strategic situation the powerful effect of information, or its absence, emerges. The tightly integrated nuclear command system will break up into separate islands of forces, each isolated from the other.... There will not be *one* assessment [of the strategic situation], in which the president looks over the situation and decides whether to retaliate, but *several*, each performed independently by isolated forces cut off from one another. Assessment then devolves downward in the command organization and decentralizes to the local commanders in charge of the separated islands [components of the graph].'

The risk of breakdown in military communications is not new. What is essentially novel about modern nuclear weapons and delivery systems is that each local commander, e.g. of a single missile-carrying submarine, individually possesses the means of inflicting devastation that no isolated prenuclear armies, navies, and air forces could inflict.

If the command system were no longer connected (in the graph-theoretic sense) after a nuclear attack, the commander of each isolated component might react to the attack without knowing the responses of other components. Even if the National Command Authorities wished to limit further hostilities, they would have lost control of some, say PAC, forces. Fearing that PAC may continue to attack and that the Soviet Union may react to the PAC attacks by attacking the NCA or the rest of the United States, the NCA may see no incentive to limit further hostilities ([**5**], pp. 126–128). As a consequence, a loss of connectivity in the command system makes conflict highly unstable.

Further, if the connectivity of the United States' command system is destroyed, there is no single force that the Soviet Union can negotiate with, and conversely if the connectivity of the Soviet Union's command system is destroyed. Therefore there is at least some incentive for each side to protect, or at least not seek actively to disrupt, the connectivity of the opposing side's command system.

Steinbruner ([**32**], pp. 43–44) argues, on the contrary, that in the face of an impending atomic attack each side's own 'command-system vulnerability presents a much more powerful incentive to initiate attack before damage has actually been suffered, an incentive that is driven...by practical fears of decisive defeat in a war that cannot be avoided'.

Whether desiring to preserve or to destroy its or the opposing command system, each side has an interest in knowing how to estimate the connectivity of the command systems of both sides. The physical means of strategic communication, their vulnerabilities and prospects, are reviewed by Carter [**9**]. Of course, understanding connectivity is only a first step toward understanding the dynamics of strategic performance, for which much more specific and sensitive measures are required (e.g. Steinbruner [**32**]).

The probability that the NCA can communicate (issue orders, receive information, or both) with all its forces, directly or by means of intermediate links, is one measure of the *reliability* of the graph of the command system. The expected number of different ways of linking all the forces (the expected number of spanning trees) is a measure of the *redundancy* of the graph of the command system. Given a matrix of dart or edge probabilities and the assumption of independence between the survival of any edge or dart and the survival of any other, Section 2 shows one way to compute the reliability of the command system, while Section 3 shows one way to compute its redundancy. Section 4 shows unexpectedly that reliability and redundancy, so measured, may conflict. For a given sum of probabilities of edges in a random graph, the allocation of probability that maximizes redundancy (namely, an isotropic allocation) may not maximize reliability, and vice versa.

6. *Open questions*

Several open questions arise naturally from the mathematics and from the preceding interpretation of it.

What is the probability that a random digraph is strongly connected? Equivalently, in a random digraph, what is the probability that, for every pair of distinct vertices i and j, there is a path with tail vertex i and head vertex j?

What is the analogue of Theorem 3 for digraphs? Specifically, among all dart probability matrices P with a given sum of probabilities $\operatorname{Tr} K(P)$, does there exist a distinguished dart probability matrix P^* for which the expected number of outtrees from vertex 1 is maximized? If so, what is P^* and is the maximum attained only at P^*?

For random graphs or random digraphs in which the sum of the probabilities (of edges or darts) is too small to permit the construction of a spanning (ordinary, in-, out-, or bi-) tree with probability 1, what is the allocation of the given sum of probabilities that maximizes reliability, measured by $P_1(V_1)$?

What are the sensitivities or 'importance factors' (Buzacott[7], p. 323) for the reliability $\partial P_1(V_1)/\partial p_{ij}$ and for the redundancy $\partial E(T)/\partial p_{ij}$ of random graphs and random digraphs? How do the sensitivities for reliability and redundancy compare? The sensitivities indicate which probabilities p_{ij} need to be measured with greater or lesser precision when estimating the reliability or redundancy of a particular network.

The problem of designing a random graph or random digraph to maximize reliability or redundancy can be made more realistic, at the cost of increased complexity of the mathematics. For example, in random graphs, suppose that for each pair i, j of vertices there is a cost function $f_{ij}(p)$ such that the cost of assuring an edge between i and j with probability $p_{ij} = p$ is $f_{ij}(p)$. As a first approximation, one might take $f_{ij}(p) = a_{ij} + b_{ij}p$, the sum of a fixed cost and a linearly increasing cost. Given the matrices $A = (a_{ij})$ and $B = (b_{ij})$ of cost coefficients and a total budget D, find the matrix P that maximizes reliability, or redundancy, or some convex combination of the two, subject to the budget constraint $\sum f_{ij}(p_{ij}) \leqslant D$ and the natural bounds $0 \leqslant p_{ij} \leqslant 1$.

For random graphs or digraphs with a large but finite number of vertices, it may be difficult to estimate each p_{ij} from believable data or theory. In such cases, it would be of interest to investigate a doubly stochastic model in which each p_{ij} is first drawn from some distribution, e.g. beta with given parameters, and then a random graph or digraph is drawn from the distribution specified by $P = (p_{ij})$.

If the edge or dart probabilities were chosen from a beta or other parametric family of distributions, it would be interesting to investigate how the parameters should be scaled as the number of vertices increases so that the reliability or redundancy approach a limit, and to determine how the reliability or redundancy depend on the parameters in the limit of a large number of vertices.

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