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Long-Run Growth Rates of Discrete Multiplicative Processes in Markovian Environments

JOEL E. COHEN

The Rockefeller University, 1230 York Avenue, New York, New York 10021

Submitted by G. Birkhoff

If the matrix of parameters of a discrete multiplicative process is subject to certain sequentially dependent random perturbations, the long-run growth rate μ of the average process is not in general bounded above by the largest $\sup_i \lambda_i$ of the growth rates λ_i of the individual matrices which drive the process. The growth rate μ of the average process may, in general, be greater or less than the long-run growth rate λ^* of a deterministic process governed by the time-averaged matrix. The time-averaged matrix may suggest that the process will be critical or subcritical ($\lambda^* \leq 1$), whereas the sequential dependence among perturbations may actually make the average process supercritical ($\mu > 1$). This suggests that in collecting data on and analyzing the dynamics of randomly perturbed discrete multiplicative processes, it is necessary to consider possible sequential dependence among matrix parameters in addition to their relative frequencies and average values. Applications to nuclear reactors, age-structured populations and other areas are indicated.

INTRODUCTION

Discrete multiplicative processes have been used in multigroup diffusion models for nuclear reactors [1, 2], in models of age-structured biological (including human) populations [3], and in other areas. The analysis of these processes has often assumed a time-independent environment. A heuristic justification for this assumption might be that if the true matrix coefficients which formally represent the environment in a multiplicative process vary according to a stationary stochastic process, it appears reasonable to replace the variable matrices by their time-average.

The purpose of this paper is to point out some consequences of the possibility that the environment may be subject to sequentially dependent, specifically Markovian, random perturbations. In particular, suppose that a process experiences a finite number $s > 1$ of environments, each represented by a corresponding matrix. Let the long-run rate of growth per discrete time unit in number of individuals (such as neutrons or people) be λ_i if the process constantly experiences the ith environment only. The process is called subcritical in

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environment i if $\lambda_i < 1$, critical if $\lambda_i = 1$, and supercritical if $\lambda_i > 1$. Suppose $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_s$, and suppose successive environments are chosen by an s-state irreducible aperiodic homogeneous first-order Markov chain, perhaps the simplest model of a sequentially dependent process. Suppose the transition probability matrix of the Markov chain is $t = (t_{ij})$.

We shall show first that a knowledge of t and λ_i, $i = 1,..., s$ does not in general provide any upper bound on the long-run growth rate μ of the expected process with Markovian environments, by constructing a 2-state process with given irreducible aperiodic t and given λ_1, λ_2 such that μ is unboundedly large. However, given t and λ_i, μ is bounded below by $\sup_{i=1,...,s} t_{ii}\lambda_i$.

We shall also show that the time-averaged matrix may imply a long-run growth rate λ^* that is arbitrarily close to λ_s at the same time that μ is arbitrarily close to λ_1. Thus the time-averaged environment may predict a critical or subcritical process while the sequential dependence among environments may render the average process supercritical.

Background

An array (vector or matrix) is nonnegative if each element is nonnegative; it is positive if each element is positive. A discrete multiplicative process with a finite number k of groups is a system of difference equations of the form

$$y(n + 1) = x(n + 1)\, y(n), \qquad n = 0, 1,... \tag{1}$$

where $y(\cdot)$ is a nonnegative k-element column vector and $x(\cdot)$ is a nonnegative $k \times k$ matrix. In applications (below), the elements of the vector $y(n)$ specify the number of individuals of some kind in each of k groups at time n; and $x_{ij}(n + 1)$ denotes the expected number of individuals in group i at time $n + 1$ resulting from a single individual in group j at time n.

A random discrete multiplicative process, studied for example in [4], is a system of random nonnegative k-vectors $Y(n, \omega)$ and random $k \times k$ nonnegative matrices $X(n, \omega)$ which satisfy

$$Y(n + 1, \omega) = X(n + 1, \omega)\, Y(n, \omega), \qquad n = 0, 1,... \tag{2}$$

for each realization ω of the process, that is, for each point ω in some underlying probability space. We shall drop ω henceforth. We assume $Y(0)$ is positive almost surely.

The matrices $X(n)$ may be viewed as the expected value matrices of the closely related processes known as multi-type branching processes in random environments [5, 6, 7].

Let $X = \{x^{(1)}, x^{(2)},..., x^{(s)}\}$ be a set of $k \times k$ nonnegative matrices with no zero columns and no zero rows such that any product of a fixed number $g \geqslant 1$

of factors which are members of X is positive; and for each x in X, $\min^+(x)/\max(x) > r > 0$, where $\min^+(x)$ is the smallest of the positive elements of x, $\max(x)$ is the largest element of x, and $r > 0$ is independent of x. X is called an "ergodic set" [8].

Let $\{X(n)\}$ be an irreducible aperiodic homogeneous Markov chain with state space X and with *column*-stochastic transition probability matrix $t = (t_{ij})$:

$$\mathbf{P}[X(n+1) = x^{(i)} \mid X(n) = x^{(j)}] = t_{ij}, \qquad i, j = 1,\ldots, s. \tag{3}$$

We call the system (2) and (3), specified by X and t, a discrete multiplicative process in Markovian environments. Henceforth we consider only this special case of (2).

There are two natural measures of the long-run rate of growth per unit time of the system (2) and (3). One is the average of the growth rates of the individual sample paths. The other is the growth rate of the average of the sample paths. Denote the first by λ, the second by μ. We shall describe procedures for calculating λ and μ, and then present new properties of μ.

It follows from results in [4] that λ is positive and finite and

$$\begin{aligned} \log \lambda &= \lim_{n\to\infty} n^{-1} E[\log Y_i(n)] \\ &= \lim_{n\to\infty} n^{-1} \log Y_i(n) \text{ almost surely}, \qquad i = 1,\ldots, k. \end{aligned} \tag{4}$$

The right equality in (4) justifies the claim that a discrete multiplicative process in Markovian environments has a long-run almost sure growth rate λ per unit time. This λ is the geometric mean of the rates of increase per unit time of a sample path. It is natural to say that the process is almost surely subcritical if $\lambda < 1$, critical if $\lambda = 1$, and supercritical if $\lambda > 1$.

To calculate λ, one first calculates the limiting joint distribution F of matrices $X(n)$ and normalized vectors $Y(n)/\| Y(n)\|$, where $\| y \| = \Sigma_i \mid y_i \mid$ for any real vector y. Recall that X is defined as the set of possible values of $X(n)$. Define Y as the set of all nonnegative k-vectors y such that $\| y \| = 1$. Let $Z = X \times Y$. Then for (x, y) in Z, $F(dx, dy)$ is the long-run probability density, roughly speaking, of $X(n) = x$ and $Y(n)/\| Y(n)\| = y$. F is the solution of the linear integral equation in [9, p. 22, Theorem 3(ii)]. A numerical example of the calculation of F is given in [10], where the integral equation is approximated by a large system of linear equations which are solved iteratively.

Given F, one obtains λ from

$$\log \lambda = \int_{(x,y)\in Z} \int_{x'\in X} \log(\| x'y \|/\| y \|)\, \mathbf{P}[X(n+1) = x' \mid X(n) = x]\, F(dx, dy). \tag{5}$$

Of course, the denominator $\| y \|$ in (5) is redundant since $\| y \| = 1$ if $y \in Y$. Harry Kesten (personal communication) pointed out the need for the log under

the integrals on the right of (5). (If log is omitted from both sides of (5), as in [9, p. 36], the resulting quantity is the arithmetic mean of the rates of increase per unit time of a sample path. We reserve λ henceforth here for the quantity in (4) and (5).)

Since $\| xy \| = \sum_{j=1}^{k} y_j \cdot (j\text{th column sum of } x)$, it follows from (5) that $\log \lambda$ is bounded above and below by

$$-\infty < \sum_{i=1}^{s} \pi_i \log c_i \leqslant \log \lambda \leqslant \sum_{i=1}^{s} \pi_i \log \| x^{(i)} \| < \infty$$

where c_i is the smallest column sum of the matrix $x^{(i)}$. As usual π_i is the equilibrium probability of $x^{(i)}$ ($t\pi = \pi$) and $\| x^{(i)} \| = \sup\{\| x^{(i)}y \| \mid \| y \| = 1\}$ = the largest column sum of $x^{(i)}$. By definition of an ergodic set, each $x^{(i)}$ has no zero columns, so $c_i > 0$. These bounds are the best possible in the sense that there exist ergodic sets X such that, for any primitive column-stochastic transition probability matrix t, the upper and lower bounds on $\log \lambda$ are equal. Specifically, let v_i, $i = 1,..., s$ be any s positive k-vectors, and let $x^{(i)}$ have each column equal to v_i. Then for every k-vector y such that $\| y \| = 1$, $c_i = \| x^{(i)}y \| = \| x^{(i)} \| = \| v_i \|$. With any such ergodic set X, $\log \lambda = \sum_{i=1}^{s} \pi_i \log \lambda_i$.

Let $\rho(\cdot)$ be the spectral radius, or maximum of the moduli of the eigenvalues, of a square matrix argument. Then the growth rate μ of the average sample path satisfies

$$\log \mu = \lim_{n \to \infty} n^{-1} \log E[Y_i(n)] \tag{6}$$

and is given [10][1] by $\mu = \rho(t \otimes X)$ where $t \otimes X$ is a nonnegative matrix of order $(ks) \times (ks)$ containing s^2 blocks of elements, each block of order $k \times k$. The (i, j)th block is the matrix $t_{ij}x^{(i)}$. Unfortunately λ was also used to denote $\rho(t \otimes X)$ in [10], where λ and μ were mistakenly taken as identical. Since log is concave upward, $\mu \geqslant \lambda$ (where λ is given by (4) and (5)) with strict inequality in general. This inequality is well known for scalar processes ($k = 1$) [11].

Results

The results of this paper concern properties of μ. Let $\lambda_i = \rho(x^{(i)})$, $i = 1,..., s$. Given t and λ_i, $i = 1,..., s$, μ can be arbitrarily large.

[1] I thank Harry Kesten for pointing out that the last four lines of page 466 require correction. They should read:

(iv) For all i and j, with probability 1, $\lim_n n^{-1} \log[L_n(i, j)] < \log \lambda$, with strict inequality in general.

The almost sure limit in (iv) is proved to exist and denoted as E by Furstenberg and Kesten ((1960), corollary on p. 462). Our Corollary 1 gives an easy explicit means of calculating an upper bound on that limit.

THEOREM 1. *Let t be the irreducible aperiodic transition matrix of a 2-state Markov chain, and let $\lambda_1 > 0$, $\lambda_2 > 0$ be given. For any B, $0 < B < \infty$, there exists an ergodic set $X = \{x^{(1)}, x^{(2)}\}$ such that $\lambda_i = \rho(x^{(i)})$, $i = 1, 2$, and $\mu = \rho(t \otimes X) > B$.*

Proof. Since $\rho(A^2) = (\rho(A))^2$ if $A \geqslant 0$ is primitive, and since $t \otimes X$ is primitive [10], it suffices to construct X such that $\rho([t \otimes X]^2) > B^2$. Let

$$x^{(1)} = \lambda_1 \begin{pmatrix} \frac{1}{2} & 2B^2/(\lambda_1\lambda_2 t_{12}t_{21}) \\ \lambda_1\lambda_2 t_{12}t_{21}/(4B^2) & 0 \end{pmatrix}$$

$$x^{(2)} = \lambda_2 \begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}.$$

$x^{(1)}$ is well-defined since the irreducibility of t implies that $t_{12}t_{21} > 0$. It is readily checked that $\rho(x^{(i)}) = \lambda_i$, $i = 1, 2$. The matrix $t \otimes X$ is of order 4×4. The upper left 2×2 submatrix of $(t \otimes X)^2$ is $t_{11}^2(x^{(1)})^2 + t_{12}t_{21}x^{(1)}x^{(2)}$. The $_{11}$ element of the second term $t_{12}t_{21}x^{(1)}x^{(2)}$ is $\lambda_1\lambda_2 t_{12}t_{21}(\frac{1}{4} + B^2/(\lambda_1\lambda_2 t_{12}t_{21})) > B^2$. Hence the $_{11}$ element of $(t \otimes X)^2$ is greater than B^2. Since the spectral radius of a nonnegative primitive matrix exceeds every diagonal element, $\mu^2 > B^2$. Because $\mu > 0$, $\mu > B$, as desired.

THEOREM 2. *For any discrete multiplicative process in Markovian environments with irreducible aperiodic t and ergodic set X, $\mu \geqslant \sup_{i=1,\ldots,s} t_{ii}\lambda_i$.*

Proof. $t \otimes X$ is primitive, so its right eigenvector α corresponding to μ is positive. Let $\alpha[i]$ refer to the ith block of k (positive) elements in α, i.e., for $i = 1,\ldots, s$, $\alpha[i]$ is a k-vector with elements $\alpha[i]^T = (\alpha((i-1)k+1), \alpha((i-1)k+2),\ldots, \alpha(ik))$. Then $\mu\alpha = (t \otimes X)\alpha$ means, for $i = 1,\ldots, s$, $\mu\alpha[i] = \sum_{j=1}^{s} x^{(i)}t_{ij}\alpha[j] = x^{(i)}t_{ii}\alpha[i] + x^{(i)}\sum_{j\neq i} t_{ij}\alpha[j]$. Thus $\mu\alpha[i] \geqslant x^{(i)}t_{ii}\alpha[i]$. Therefore μ is an upper bound for $\rho(x^{(i)}t_{ii}) = t_{ii}\lambda_i$. Since this is true for all $i = 1,\ldots, s$, the theorem is proved.

Let the stationary probability vector of the Markov chain on X be π, a positive column s-vector satisfying $t\pi = \pi$. Then, since $\lim_{n\to\infty}\mathbf{P}[X(n) = x^{(i)}] = \pi_i$, $i = 1,\ldots, s$, the long-run average environment corresponds to the average matrix

$$x^* = \sum_{i=1}^{s} \pi_i x^{(i)}. \tag{7}$$

Let $\lambda^* = \rho(x^*)$. In general, if not all λ_i are equal, then for some t not of rank 1, $\mu = \rho(t \otimes X) > \lambda^*$ and for other such t, $\mu < \lambda^*$.

When there is no sequential dependence among the s environments, t is a matrix of rank 1 with each column equal to π. In this case of sequentially independent perturbations, $\rho(t \otimes X) = \lambda^*$ [10] and the replacement of (2) by (1) with $x(n) = x^*$ gives the long-run growth rate of the average process. According

to Harry Kesten (personal communication), in this case Paulson and Uppuluri in an unpublished manuscript also calculated moments of $Y(n)$ using the usual Kronecker product.

THEOREM 3. *Assume the elements $x^{(i)}$ of X are labelled so that $\lambda_1 \geq \cdots \geq \lambda_s$. Given any positive probability s-vector π and any $\epsilon > 0$, there exists an irreducible aperiodic homogeneous Markov chain with state space X and a transition probability matrix t satisfying $t\pi = \pi$ such that $| \mu - \lambda_1 | < \epsilon$.*

Proof. For ϕ satisfying $0 < \phi < \min(\pi_1, \ldots, \pi_s)$, define $t(\phi)$ by $t_{ii} = 1 - \phi/\pi_i$, $i = 1, \ldots, s$; $t_{i+1,i} = \phi/\pi_i$, $i = 1, \ldots, s-1$; $t_{1s} = \phi/\pi_s$; and $t_{ij} = 0$ otherwise. Then $t(\phi)$ is the column-stochastic transition probability matrix of an s-state irreducible aperiodic homogeneous Markov chain and $t(\phi)\,\pi = \pi$. Since $\lim_{\phi \downarrow 0} t(\phi) = I$, where I is an $s \times s$ identity matrix, $\lim_{\phi \downarrow 0} \rho(t(\phi) \otimes X) = \rho(I \otimes X)$ by continuity [12]. But $I \otimes X$ is a block-diagonal matrix with $x^{(i)}$ along the principal diagonal, so $\rho(I \otimes X) = \lambda_1$ [13]. Thus for any $\epsilon > 0$, ϕ can be chosen small enough that with $\mu = \rho(t(\phi) \otimes X)$, $| \mu - \lambda_1 | < \epsilon$.

THEOREM 4. *For any $\epsilon > 0$, there exists an irreducible aperiodic homogeneous Markov chain with state space X such that $| \lambda^* - \lambda_s | < \epsilon$ and $| \mu - \lambda_1 | < \epsilon$.*

Proof. For $m = s, s+1, \ldots$, let $\pi(m)$ be the probability s-vector in which the first $s-1$ elements equal $1/m$ and the sth element equals $1 - (s-1)/m$. Then as m increases without limit, $\pi(m)$ approaches the s-vector with sth element equal to 1 and 0 elsewhere. Thus $\lim_{m \to \infty} \sum_{i=1}^{s} (\pi(m))_i\, x^{(i)} = x^{(s)}$, so again by continuity [12], $\lim_{m \to \infty} \rho(\sum_{i=1}^{s} (\pi(m))_i\, x^{(i)}) = \lambda_s$. Let π be any vector $\pi(m)$ such that $| \rho(\sum_{i=1}^{s} (\pi(m))_i\, x^{(i)}) - \lambda_s | < \epsilon$. For this value of π and the given ϵ, construct the required Markov chain using Theorem 3.

Theorem 4 obviously also holds when λ_s is replaced by λ_i for any fixed $i = 1, \ldots, s$.

APPLICATIONS

In iterative computational solutions of multigroup diffusion models for nuclear reactors [2, 14], the elements of the vector $y(n)$ specify the number of neutrons in each of k groups, commonly the so-called space and lethargy (or energy) groups, at time n; and $x_{ij}(n+1)$ denotes the expected number of prompt neutrons in group i at time $n+1$ produced by a single neutron in group j at time n. Most steady-state calculations for a reactor operating at power treat $x(n)$ as independent of time n. Under this assumption, criticality of the reactor is determined by $\rho(x(n))$.

However, reactors are physically subject to sequentially dependent random perturbations. These perturbations include fluctuations in the distribution of

volatile isotopes of bromine and iodine [14, p. 465], vibrations of control rods [14, p. 474], fluctuations in density associated with bubble formation in a boiling water reactor [14, p. 513], bowing of fuel rods towards the center of a reactor and mechanical motions of the plate supporting the fuel rods [14, p. 515], and localized variations in the production of xenon–135 from iodine–135 [14, p. 555]. Depending on the scale of variations in space and time, these perturbations have been modeled by piecewise static processes, sinusoidal oscillations, and other approximations. The results presented here suggest the merit of addressing directly the patterns of sequentially dependent random fluctuations and provide some general techniques for doing so.

In models of age-structured biological populations, the elements of the vector $y(n)$ specify the number of individuals in each of k age groups; and $x_{ij}(n+1)$ denotes the expected number of offspring or of survivors in group i at time $n+1$ due to a single individual in group j at time n. In applications of this model to human populations, Sykes [15] treated $x(n)$ as a random matrix with no sequential dependence of $x(n+1)$ on $x(n)$. This model, with an allowance for random immigration also having no sequential dependence, has now been extensively studied [16]. Others [17, 18, 19] have described the demographic and biological interpretation of a Markovian dependence of $x(n+1)$ on $x(n)$. A nonlinear age-structured model in Markovian environments has been applied in detail to mallard populations [18]. The empirical usefulness for human populations of the model (2) and (3) is under study.

The functioning of an automaton in a stationary random medium is described by a Markov chain in which the vector $y(n)$ specifies the probability distribution of the states of the automaton at time n, and the time-homogeneous column-stochastic matrix $x(n+1)$ specifies the state transition probabilities, which depend on the action of the automaton at time n and the stochastic response of the medium at $n+1$ [20, pp. 12–16]. The probabilities governing the response of the medium are stationary in time. In an automaton in a composite random medium, the probabilities governing the response of the medium are chosen by a Markov chain from a finite set of arrays of response probabilities [20, pp. 25–31]. This model of an automaton in a composite random medium leads to equations which are a special case of (2) and (3) in which each matrix in the set X is column-stochastic. It is possible to recognize in Eq. 20 of [20, p. 26] a special case of the generalized tensor product denoted by $\otimes$ here and in [10].

A model for ecological or biochemical material cycling systems which is identical to an automaton in a composite random medium has been proposed [21], but the analysis of the model is seriously wrong.

The results proved apply to these processes and to any others for which a discrete multiplicative process in Markovian environments can serve as an appropriate model. If the process is supercritical in even one environment which occurs with positive probability, no matter how small, then sequential dependence among environments can render the average of sample paths supercritical

without altering the time-average of the environments. This finding suggests that in collecting data on and analyzing the dynamics of randomly perturbed multiplicative processes, it is crucial to consider possible sequential dependence among environments in addition to their relative frequencies and average values.

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