Further Properties of Third Order Determinants

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FURTHER PROPERTIES OF THIRD ORDER DETERMINANTS

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C. W. Trigg (MATHEMATICS MAGAZINE 35: 78, 1962) presents the following property of third order determinants and their related “twists:”

\[
\begin{vmatrix}
  a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 
\end{vmatrix}
+ \begin{vmatrix}
  c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 
\end{vmatrix}
+ \begin{vmatrix}
  b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 
\end{vmatrix} = 0.
\]

A different, but equivalent, representation of this result reveals an underlying symmetry which suggests two further analogous properties.

Let \( V_i = (a_i, b_i, c_i), i = 1, 2, 3 \). Define \( s^0 V_i = V_i, s^1 V_i = (c_i, a_i, b_i), s^2 V_i = (b_i, c_i, a_i) \). Then (1) takes the form

\[
V_1 \cdot (V_2 \times V_3) + V_1 \cdot (s^1 V_2 \times s^2 V_3) + V_1 \cdot (s^2 V_2 \times s^1 V_3) = 0,
\]

where “.” means scalar product and “\( \times \)” means vector product. The left member of (2) equals \( V_1 \cdot (V_2 \times V_3 + s^1 V_2 \times s^2 V_3 + s^2 V_2 \times s^1 V_3) \), and the right hand factor of this product must be 0, since (2) holds for all \( V_1 \). For arbitrary \( (V_2, V_3) \), define

\[
[s^0 \times s^0 + s^1 \times s^2 + s^2 \times s^1](V_2, V_3) = V_2 \times V_3 + s^1 V_2 \times s^2 V_3 + s^2 V_2 \times s^1 V_3.
\]

Then

\[
s^1 \times s^2 + s^0 \times s^0 + s^2 \times s^1 = 0,
\]

i.e., the left member annihilates every pair of 3-vectors. The form of (3) suggests that

\[
s^0 \times s^2 + s^1 \times s^1 + s^2 \times s^0 = 0
\]

and

\[
s^0 \times s^1 + s^2 \times s^2 + s^1 \times s^0 = 0,
\]

both of which may be easily verified.

These results also hold for vertical permutations. If \( *V \) is the column vector which is the transpose of \( V \), and \( W \) is a column 3-vector, define \( tW = *s^{-1}W \). Then we may substitute \( t \) for \( s \) in (3), (4), and (5), since the value of a determinant is not affected by interchanging rows and columns.