

Some Relationships Between Music and Mathematics

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INTUITIVE AWARENESS of a relationship between music and mathematics existed as early as Pythagoras.

Pythagoras made an attempt to investigate and determine this relationship through acoustics. By varying the vibrating length of the string on a monochord, he discovered that the ratios of the lengths of strings whose vibrations produce consonant musical intervals may be expressed by small integers. But he departed from physical description when he attributed the quality called consonance to the presence of simple number ratios.¹

Now it appears that the theory that consonance is solely related to simple integer ratios is not entirely correct. Slightly mistuned consonances remain consonant though their ratios become very complex.² Further, the concept of consonance seems to have evolved. For instance, in the parallel organum of the ninth and tenth centuries, the duplum accompanies the tenor in the lower fifth or fourth, note against note. By the thirteenth century, an English form of two-part polyphony called gymel was sung almost entirely in parallel thirds.³ The major and minor third were not considered consonances by the Pythagoreans. In addition, the fourth was not considered admissible in the first species counterpoint of the sixteenth century between the bottom two voices of three-part counterpoint.

The real Pythagorean contribution lies in the intuition which grouped music with mathematics. The Pythagoreans founded the *quadrivium*, the fourfold way to knowledge. They divided mathe-

matical science into two parts: one, "how many"; the other, "how much." They divided each of these parts into two. "They said that discrete quantity or the 'how many' is either absolute or relative, and that continued quantity or the 'how much' is either stable or in motion. Hence they laid down that arithmetic contemplates that discrete quantity which subsists by itself, but music that which is related to another; and geometry considers continued quantity so far as it is immovable, but that astronomy contemplates continued quantity so far as it is of a self-motive nature."⁴

This *quadrivium* became part of the secular curriculum taught in the medieval *scholae*, the church schools founded by Charlemagne in the eighth century. Mathematics remained grouped with music until the Renaissance, when a division which now appears temporary was imposed by extraneous circumstances.⁵

Supported and attacked by mathematicians and musicians, the intuition of a kinship between music and mathematics has persisted to today. Stravinsky speaks of "the higher mathematics of music."⁶ Most of us can think of some acquaintance whose interests include music and mathematics; if not, we may be reminded of the Oak Ridge research laboratories' symphony orchestra, composed of mathematicians and nuclear physicists. We may have wondered about the kinship of music and mathematics that gives them a common appeal.

LET US LOOK at three aspects of mathematics and music and examine their claimed affinity.

First, consider the media of music and mathematics.

The acoustician tells us that the medium of music is sound which has been highly refined and organized, sound which has been abstracted from the chaos of general sound. When seen on an oscilloscope, a musical tone approximates a sine wave. It has a definite frequency of vibration and a sustained overtone structure. Seen on an oscilloscope, a noise presents a jagged, irregular picture. "Beyond certain elementary facts of acoustics modern music shows no direct connection with nature independently of art; indeed, it is already art that determines the selection of these elementary acoustic facts . . . The ordinary non-artistic experience of sound has so little in common with music that musical realism is, with rare though popular exceptions, generally regarded as an eccentricity."⁷

Similarly, mathematics deals in the abstract and the organized. We can easily see one hand, one piece of paper, one table, but the concept of the number one is abstracted from all individual cases and is not represented by any of them. In fact, mathematically, the concept "one" is defined completely in terms of marks on paper and operations on those marks; we assign these marks a familiar and convenient intuitive meaning in order to make it easier to manipulate them. More abstract are the "literal" numbers of algebra, which refer to the set of numbers having certain required properties. The theory of sets is abstracted from these and many similar specific cases; at the height of abstraction and at the base of mathematics is symbolic or formal logic.

Bertrand Russell has defined mathematics as the science in which we never know what we are talking about. "Since, then, mathematics is an entirely free activity, unconditioned by the external world, it is

⁷D. F. Tovey, "Music," *Encyclopedia Britannica*, 11th edition, 1911, 19:72.

¹Norman Cazden, "Musical Intervals and Simple Number Ratios," *Journal of Research in Music Education*, 7(2):197, Fall 1959.

²Joseph Peterson, "A Functional View of Consonance," *Psychological Review*, 32:17, 1925.

³Willi Apel, "Gymel," *Harvard Dictionary of Music* (Harvard U. Press, Cambridge, 1953) p. 315.

⁴George J. Allman, "Pythagoras," *Encyclopedia Britannica*, 11th edition, 1911, 22:700.

⁵Henry Margenau, *The Nature of Physical Reality* (McGraw-Hill, New York, 1950), p. 16-17: "Science vs. Humanities."

⁶Igor Stravinsky, *Poetics of Music* (Vintage Books, New York, 1956) p. 47.

more just to call it an art than a science. It is as independent as music of the external world; and although, unlike music, it can be used to illuminate natural phenomena, it is just as 'subjective,' just as much a product of the free creative imagination."⁸

THE PERSISTENT BELIEF that music and mathematics are related because they both deal in relationships among small whole numbers is unfortunately wrong. If it were right, all music beyond simple triads in just intonation and all mathematics beyond fifth-grade arithmetic would have to be ignored.

Rather, music and mathematics are similar because they are systems of symbols that do not point to anything; the sounds in the air and the marks on paper refer only to themselves, their antecedents, and their successors. Basically, all mathematics, once conceived, can be and has been reduced to rules for manipulating marks on paper.⁹ Similarly, "music consists of successions and forms of sound, and these alone constitute the subject . . . music speaks not only by means of sounds, it speaks nothing but sound."¹⁰ The media of music and mathematics are similar because they are both systems of symbols referring only to themselves.

Second, consider the creative processes in music and mathematics. Hiller and Isaacson recognize five basic principles of composition. "The first principle is that the formation of a piece of music is an ordering process in which specified musical elements are selected and arranged from an infinite variety of possibilities, i.e., from *chaos*."¹¹ Stravinsky also speaks, but in more intuitive terms, of "the need we feel to bring order out of chaos, to extricate the straight line of our operation from the tangle of possibilities and the indecision of vague thoughts.

⁸John W. N. Sullivan, "Mathematics as an Art," *World of Mathematics*, ed. J. R. Newman (Simon and Schuster, New York, 1956) 3:2015-2021.

⁹N. Bourbaki, *Éléments de Mathématique, Les Structures Fondamentales de l'Analyse, Livre I: Théorie des Ensembles, Chapitre I: Description de la mathématique formelle* (Hermann, Paris, 1954).

¹⁰Eduard Hanslick, *The Beautiful in Music*, trans. G. Cohen (Liberal Arts Press, New York, 1957) p. 119.

¹¹L. A. Hiller, Jr. and L. M. Isaacson, *Experimental Music* (McGraw-Hill, New York, 1959) p. 16.

. . ."¹² The second principle is that "the degree of imposed order is itself a significant variable. The third principle is that the two most important dimensions of music upon which a greater or lesser degree of order can be imposed are pitch and time . . . the fourth principle is that memory, as well as instantaneous perception, is required in the understanding of musical structure." The fifth principle is that tonality "be considered the result of establishing pitch order in terms of memory recall"; that is, that tonality results from comparing each note or harmony with a tonic through memory.

What is mathematical creation? "To create consists precisely in not making useless combinations and in making those which are useful and which are only a small minority. Invention is discernment, choice. . . . The sterile combinations do not even present themselves to the mind of the inventor."¹³ (It is interesting to note that Stravinsky insisted he was an "inventor" of music, not a composer.)

Thus new ideas in mathematics and music are arrived at similarly, that is, by selection from a multitude of possibilities. What determines which possibilities are selected is a most significant similarity between mathematics and music.

The mathematician is as much concerned with beauty as the musician. The literature of mathematics is sprinkled with aesthetic terms; the mathematician is more concerned with finding an elegant method and with the beauty of his insights than with applications of the results. Herman Weyl, who "alone could stand comparison with the last great universal mathematicians of the nineteenth century, Hilbert and Poincaré said, "My work always tried to unite the true with the beautiful; but when I had to choose one or the other, I usually chose the beautiful."¹⁴ The mathematician finds that the beautiful usually leads to the true. He constantly searches for the most elegant proof, that is, the shortest logical line between given hypothesis and

¹²Stravinsky, *op. cit.*, p. 6.

¹³Henri Poincaré, "Mathematical Creation," *The Creative Process*, ed. Brewster Ghiselin (New American Library, New York, 1955) p. 35-36.

¹⁴Freeman J. Dyson, obituary of Hermann Weyl, *Nature*, March 10, 1956, quoted in "Commentary on Hermann Weyl," *World of Mathematics*, 3:1831.

desired conclusion. In his search for new hypotheses his intuition for the most beautiful guides him. Examples are the fundamental theorems of the integral and differential calculus, which unite in a breath-taking way some widely divergent and apparently unrelated mathematical operations.

The beauty of mathematics can be "cold and austere, like that of sculpture, without appeal to any part of our weaker nature, sublimely pure, and capable of a stern perfection as only the greatest art can show."¹⁵ It can be warm and exciting, especially when it releases tension built up in searching for it. A concise, well-proportioned proof can stimulate the same response as a Bach fugue; a daring insight and leap forward can simulate a soaring Tchaikowsky theme. Both mathematicians and musicians will recognize that this verbalization is only a gross approximation; that beauty must be experienced to be understood.

CREATION in mathematics and music only begins with new ideas. Once achieved they are exploited logically.

By using his craft, the composer consciously exploits the logical possibilities of the sounds he has set in order.¹⁶ These techniques of development can be acquired in an elementary way from classes in theory and "composition," but their manner of application distinguishes the great composer from the talented youth who can improvise at the piano. It has been said that a composition lives in its development. Yet the composer cannot develop possibilities randomly or endlessly, like a machine grinding out permutations. He must know how "to proceed by elimination . . . know how to discard."¹⁷

The composer must be consistent in his development. Schoenberg "adopted the musical system that suited his needs, and within this system, he is perfectly consistent with himself, perfectly coherent. One cannot dismiss music that he dislikes by labeling it cacophony."¹⁸

¹⁵Bertrand Russell, *The Study of Mathematics*.

¹⁶Sidney and Henry Cowell, "The Schillinger Case—Charting the Musical Range," *Modern Music*, 23(3):226, Summer 1946.

¹⁷Stravinsky, *op. cit.*, p. 70.

¹⁸*Ibid.*, p. 15.

Similarly, in mathematics, "we can start from any set of axioms we please, provided they are consistent with themselves and one another, and work out the logical consequences of them."¹⁹ Indeed, "mathematics is the science of the logically possible."²⁰ Just as logical, consistent chromaticism is as valid as the diatonic scale of Pythagorean "natural law," so non-Euclidean geometries are as valid as the logical "necessities" of Euclid.

THIRD, consider a piece of music and a mathematical demonstration.

A note or a chord in a melody or a harmonic progression acquires significance because it is related through time and memory to previous notes or chords. Isolated notes or chords have little significance; transposing a melody does not alter its musical import. The intervals between successive elements of a composition are important. "Intervals are the stuff of the composer's universe."²¹ "Music is based on temporal succession and requires alertness of memory. Consequently music is a chronologic art . . ."²²

Similarly, "a mathematical demonstration is not a simple juxtaposition of syllogisms, it is syllogisms placed in a certain order, and the order in which these elements are placed is much more important than the elements themselves."²³ The geometry of Euclid's *Elements* is not just a collection of facts, but a logical system. Each proposition follows from the preceding axioms, definitions, postulates and theorems. Euclid's main achievement was the disposition of the propositions; their logical system was the main merit of the *Elements*.²⁴ Like the third principle of musical composition, the two significant variables in mathematics are statements of logical implication and time. The arrangement of the statements in time is important.

THUS the kinship between mathematics and music has at least three aspects: (1) Both deal in similarly

¹⁹Sullivan, *op. cit.*

²⁰Philippe LeCorbeiller, "Crystals and the Future of Physics," *World of Mathematics*, 2:876.

²¹Ross Lee Finney, composer-in-residence at Univ. of Michigan, lecture at National Music Camp, Interlochen, Michigan, summer 1959.

²²Stravinsky, *op. cit.*, p. 29.

²³Poincaré, *op. cit.*, p. 35.

²⁴G. Pólva, *How to Solve It* (Doubleday and Co., Garden City, N.Y., 1957, second edition), p. 217.

abstract media unrelated to or restricted by physical data. (2) The processes of creation are similar; that is, in both the most beautiful combination is selected from an infinitude of possibilities and its logical potentials are developed consistently. (3) The end products of mathematics and music depend for their meaning upon the successive relationships of their elements and upon the order imposed by the creator.

Perhaps the easiest way to become certain of the kinship between mathematics and music is to experience the joys of creation in both. Marston Morse wrote, "Most convincing to me of the spiritual relations between mathematics and music, is my own very personal experience. Composing a little in an amateurish way, I get exactly the same elevation from a prelude that has come to me at the piano, as I do from the new idea that has come to me in mathematics."²⁵

²⁵Marston Morse, "Mathematics and the Arts," *Bulletin of the Atomic Scientists*, 15(2) :55-59, February 1959.

