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CONVEXITY OF THE DOMINANT EIGENVALUE OF AN ESSENTIALLY NONNEGATIVE MATRIX

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ABSTRACT. The dominant eigenvalue of a real $n \times n$ matrix A with nonnegative elements off the main diagonal is a convex function of the diagonal of A. We give a short proof using Trotter's product formula and a theorem on log-convexity due to Kingman.

An $n \times n$ real matrix A with nonnegative elements a_{ij} $(i \neq j)$ off the main diagonal is called essentially nonnegative. Such an A has an eigenvalue $r(A)$, called the dominant eigenvalue, that is real and greater than or equal to the real part of any other eigenvalue of A. $r(A)$ need not be the eigenvalue of A with the largest modulus, but when every element of A is nonnegative, $r(A)$ is indeed the Perron-Frobenius root or spectral radius of A. $r(A)$ was shown to be a convex function of a single diagonal element of nonnegative A, when all other elements of A are held constant, by use of determinants [1]. $r(A)$ was shown to be a convex function of all diagonal elements jointly of essentially nonnegative A by a Feynman-Kac formula from the theory of random evolutions [2] and by a variational formula of Donsker and Varadhan [3]. Here we give a short proof that $r(A)$ is a convex function of the diagonal of essentially nonnegative A using the Trotter product formula [6] and a theorem on log-convexity due to Kingman [4] as extended by Seneta [5].

The Trotter product formula for any two real $n \times n$ matrices S and T asserts

$$e^{S+T} = \lim_{k \to \infty} (e^{S/k} e^{T/k})^k. \tag{1}$$

A positive function f on a convex domain is log-convex if $\log f$ is convex on that domain. Kingman showed that the set of log-convex functions is closed under positive linear combinations, multiplication, positive powers and lim sup. Let $T(u)$ be a nonnegative (elementwise) $n \times n$ matrix in which each element is either identically 0 or is a log-convex function of a parameter u in some convex domain in R^m, $m \geq 1$. Then, provided that $r(T(u))$ is positive in this domain, $r(T(u))$ is log-convex there [5, p. 83]. (Kingman [4] assumed every element of $T(u)$ positive.)

Let D be a diagonal real $n \times n$ matrix $\operatorname{diag}(d_1, \ldots, d_n)$. Let A be essentially nonnegative.

THEOREM. *$r(A + D)$ is a convex function of D.*

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PROOF. We have $r(A + D) = \log e^{r(A+D)} = \log r(e^{A+D})$. Since r is a continuous function, (1) gives

$$r(e^{A+D}) = \lim_{k\to\infty} r\left[\left(e^{A/k}e^{D/k}\right)^k\right],$$

hence

$$r(A + D) = \lim_{k\to\infty} \log r\left[\left(e^{A/k}e^{D/k}\right)^k\right]. \tag{2}$$

Now for every positive integer k, every element of $e^{A/k}$ is nonnegative. Moreover $r(e^{A/k}) = e^{r(A/k)} > 0$, and multiplying $e^{A/k}$ by a nonnegative nonsingular matrix will yield a nonnegative matrix whose spectral radius is also positive. In particular, $T_k(D) = (e^{A/k}e^{D/k})^k$ is nonnegative with $r(T_k(D)) > 0$. For $i = 1, 2, \ldots, n$, the ith diagonal element of $e^{D/k}$, which is just $e^{d_i/k}$, is a log-convex function of D. Hence every element of $T_k(D)$ is either 0 or a log-convex function of D. The theorem of Kingman as extended by Seneta implies that $\log r(T_k(D))$ is a convex function of D. By (2), $r(A + D)$ is also a convex function of D.

Some applications of this theorem are discussed in [**1**].

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