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When Does a Leaky Compartment Model Appear to Have No Leaks?

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A finite set of first-order, linear, autonomous ordinary differential equations which conserves the total amount of some substance flowing in a system will exactly describe the distribution of that substance among observable states which leak (or suffer mortality) if and only if the rates of leakage into or out of all the states are identical.

A finite set of first-order, linear, autonomous ordinary differential equations with continuously time-varying coefficients is used to model the flow of people, particles, or probability density in a discrete state space in continuous time in epidemiology (Muench [1959]), pharmacology (Bellman [1970]), Markovian models (Hoem [1969]), and a vast variety of other applications. Such models may be written $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{x}$ is the column vector whose components give the amounts of substance in each state, the superior dot means derivative with respect to time, and $\mathbf{A}$ is a continuous square matrix function of time. Excluding empty systems, we assume with no loss in generality that $\mathbf{x}(0)^T \cdot \mathbf{1} = 1$, where $\mathbf{1}$ is a column vector of which each component is 1, and superscript T means transpose. If $\mathbf{0}$ is the column vector of which each element is 0, then the condition that the model be conservative, i.e., that the amount of substance flowing in the model be neither created nor destroyed in time, is simply $\mathbf{A}^T \cdot \mathbf{1} = \mathbf{0}$. We assume $\mathbf{A}$ is such that for $t > 0$, $\mathbf{x}(t) > \mathbf{0}$.

The phenomena described by such conservative linear models frequently have leaks: flows from the observable states considered in the model to unobservable states. In epidemiological models, people die or emigrate while the living in various states of health or parity are being counted. In pharmacological studies, particles are excreted, degraded, or otherwise escape measurement as the remaining concentrations in various organs are assayed. To model such phenomena, let $\mathbf{M}$ be a diagonal matrix whose diagonal elements measure the continuously time-varying rate of leakage between (into or out of) the observed states and an unobserved state. Assuming no losses to start with, the state vector $\mathbf{y}$ of this model satisfies $\dot{\mathbf{y}} = (\mathbf{A} - \mathbf{M})\,\mathbf{y}$, $\mathbf{y}(0) = \mathbf{x}(0)$.

Then the fractions of the observable substance which remain in each of the observable states are the components of $\mathbf{z} = \mathbf{y}/(\mathbf{y}^T \cdot \mathbf{1})$. We prove that $\mathbf{z} = \mathbf{x}$ if and only if $\mathbf{M} = m\mathbf{I}$, where m is a continuously time-varying scalar and $\mathbf{I}$ is the identity matrix. The point of the theorem is that if mortality or leakage rates differ from one state to another, then estimates of the rates of flow $\mathbf{A}$ based on the observable distributions $\mathbf{z}$ of substance among states will be biased.

In Hoem's [1969] analysis of Markov chains, $\mathbf{x}$ corresponds to the partial probability distribution when there is no mortality and $\mathbf{z}$ corresponds to the probability distribution of the observable states when transitions have been influenced by mortality. Hoem ([1969, p. 154]) established that $\mathbf{M}$ being scalar is sufficient to assure $\mathbf{z} = \mathbf{x}$, generalizing the result previously known for neutral mortality change. The present note establishes Hoem's result and, for the first time, its converse, with a new, concise proof. Whereas the proof of sufficiency does not require the assumption made above that $\mathbf{x} > \mathbf{0}$ and hence $\mathbf{z} > \mathbf{0}$, the proof of necessity does.

To prove $\mathbf{z} = \mathbf{x}$ if and only if $\mathbf{M} = m\mathbf{I}$, calculate $\dot{\mathbf{z}}$. Since

$$\dot{\mathbf{y}}^T \cdot \mathbf{1} = \mathbf{y}^T(\mathbf{A}^T - \mathbf{M}^T)\,\mathbf{1} = -\mathbf{y}^T\mathbf{M}^T\mathbf{1}$$

because $\mathbf{A}^T \cdot \mathbf{1} = \mathbf{0}$, we have

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} - \mathbf{M}\mathbf{z} + \mathbf{z}^T\mathbf{M}^T\mathbf{1}\mathbf{z}. \tag{1}$$

Now if $\mathbf{M} = m\mathbf{I}$, then $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} - m\mathbf{z} + m\mathbf{z} = \mathbf{A}\mathbf{z}$ since $\mathbf{z}^T \cdot \mathbf{1} = 1$; and since $\mathbf{z}(0) = \mathbf{x}(0)$ we must have $\mathbf{z} = \mathbf{x}$ always. Conversely, if $\mathbf{z} = \mathbf{x}$, then subtracting $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}$ from (1) implies $\mathbf{M}\mathbf{z} = (\mathbf{z}^T\mathbf{M}^T\mathbf{1})\mathbf{z}$. Since $(\mathbf{z}^T\mathbf{M}^T\mathbf{1})$ is a scalar and $\mathbf{z} > \mathbf{0}$, $\mathbf{M} = (\mathbf{z}^T\mathbf{M}^T\mathbf{1})\mathbf{I}$, as desired.

This result has at least one practical application (Cohen [1973]).

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