COMMENTS ON PAPERS AND BOOKS

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This department will present comments on papers published in the MATHEMATICS MAGAZINE, lists of new books, and reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

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FURTHER PROPERTIES OF THIRD ORDER DETERMINANTS

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C. W. Trigg (MATHEMATICS MAGAZINE 35: 78, 1962) presents the following property of third order determinants and their related "twists:"

$$(1)\qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ c_2 & a_2 & b_2 \\ b_3 & c_3 & a_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ b_2 & c_2 & a_2 \\ c_3 & a_3 & b_3 \end{vmatrix} = 0.$$

A different, but equivalent, representation of this result reveals an underlying symmetry which suggests two further analogous properties.

Let $V_i=(a_i,\ b_i,\ c_i)$, $i=1,\ 2,\ 3$. Define $s^0V_i=V_i$, $s^1V_i=(c_i,\ a_i,\ b_i)$, $s^2V_i=(b_i,\ c_i,\ a_i)\ \cdots$. Then (1) takes the form

$$(2)\qquad V_1\cdot(V_2\times V_3)+V_1\cdot(s^1V_2\times s^2V_3)+V_1\cdot(s^2V_2\times s^1V_3)=0,$$

where "$\cdot$" means scalar product and "$\times$" means vector product. The left member of (2) equals $V_1\cdot(V_2\times V_3+s^1V_2\times s^2V_3+s^2V_2\times s^1V_3)$, and the right hand factor of this product must be 0, since (2) holds for all V_1. For arbitrary $(V_2,\ V_3)$, define

$$[s^0\times s^0+s^1\times s^2+s^2\times s^1](V_2,\ V_3)=V_2\times V_3+s^1V_2\times s^2V_3+s^2V_2\times s^1V_3.$$

Then

$$(3)\qquad s^1\times s^2+s^0\times s^0+s^2\times s^1=0,$$

i.e., the left member annihilates every pair of 3-vectors. The form of (3) suggests that

$$(4)\qquad s^0\times s^2+s^1\times s^1+s^2\times s^0=0$$

and

$$(5)\qquad s^0\times s^1+s^2\times s^2+s^1\times s^0=0,$$

both of which may be easily verified.

These results also hold for vertical permutations. If $*V$ is the column vector which is the transpose of V, and W is a column 3-vector, define $tW={*s*}^{-1}W$. Then we may substitute t for s in (3), (4), and (5), since the value of a determinant is not affected by interchanging rows and columns.